Discrete Mathematics 338 (2015) 1649–1658

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Unimodality of partitions in near-rectangular Ferrers diagrams

Samuel Zbarsky

Carnegie Mellon University, United States

ARTICLE INFO

Article history:
Received 8 September 2014
Received in revised form 24 March 2015
Accepted 3 April 2015
Available online 6 May 2015

Keywords:
Integer partition
Unimodality
Generating function

ABSTRACT

We look at the rank generating function G_λ of partitions inside the Ferrers diagram of some partition λ, investigated by Stanton in 1990, as well as a closely related problem investigated by Stanley and Zanello in 2013. We show that G_λ is not unimodal for a larger class of 4-part partitions than previously known, and also that if the ratios of parts of λ are close enough to 1 (depending on how many parts λ has), or if the first part is at least half the size of λ, then G_λ is unimodal.

1. Introduction

Given partition λ with $\lambda = (\lambda_1, \dots, \lambda_b)$, the *length* of λ is b and the *size* of λ, denoted $|\lambda|$, is $\sum \lambda_i$. Given partitions μ and λ, we say that $\mu \subseteq \lambda$ if the Ferrers diagram of μ fits inside the Ferrers diagram of λ. For any partition λ, we look at the set of partitions $\mu \subseteq \lambda$, ordered by the relation above, and let G_λ be the rank-generating function of this poset (i.e. $G_\lambda(q) = \sum a_n q^n$ where a_n is the number of partitions $\mu \subseteq \lambda$ of size n). We are interested in understanding when the coefficients of G_λ are unimodal, i.e. first weakly increasing and then weakly decreasing. This question was considered by Stanton [6].

Let b be the length of λ. In the case that $\lambda = (a, \dots, a)$, we get that $G_\lambda(q)$ is the Gaussian binomial coefficient $G_\lambda(q) = \binom{a+b}{b}_q$. There are many proofs that these are unimodal symmetric (for instance, [7,4,3,8]). This fact will be used many times in the proof of our main result.

We can also look at the question above, but require that both λ and μ have distinct nonzero parts. We call this rank generating function $F_\lambda(q)$. The unimodality of $F_\lambda(q)$ was considered by Stanley and Zanello [5]. For this version, the partition $\lambda = (b, b-1, \dots, 1)$ gives the generating function

$$F_\lambda(q) = \prod_{i=1}^{b} (1 + q^b)$$

which is unimodal symmetric (see, for instance, [4]). Alpoge [1] proved that for $n \gg b$, if $\lambda = (n, n-1, \dots, n-b+1)$, then F_λ is unimodal. We will mainly be concerned with Stanton's problem, but some of our results will also extend to the Stanley–Zanello version. For both problems, it is proven by Stanton and Stanley–Zanello respectively that any partition with at most 3 parts has a unimodal generating function, and that there are infinite families of partitions with 4 parts that do not.

In Section 2, we will derive a form for F_λ and G_λ when the length of λ is fixed and show that F_λ and G_λ are unimodal if $2\lambda_1 \geq |\lambda|$. In Section 3, we will prove that for a reasonable notion of positive density, for both versions of the problem, there is a positive density of partitions with 4 parts that have nonunimodal generating functions, giving larger classes of

E-mail address: sa_zbarsky@yahoo.com.

http://dx.doi.org/10.1016/j.disc.2015.04.004

nonunimodal partitions than were known before. In Sections 4 and 5, we will prove that partitions with 5 or more parts that are near-rectangular (in a sense to be defined later in the paper) and with $|\lambda| \gg b$ have unimodal G_λ. Finally, in Section 6, we present some conjectures that are supported by the results in this paper and by computational data.

2. Generating function

We will use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. We begin by finding an expression for $G_\lambda(q)$ when λ has a fixed number of parts b.

If λ is a partition, denote by G^i_λ the size generating function for the number of partitions $\mu \subseteq \lambda$, giving the first part of the partition the weight i (for instance, the partition $(4, 2, 1)$ will contribute to the q^{15} coefficient of G^3_λ since $3(4)+2+1 = 15$). Note that $G^1_\lambda = G_\lambda$. Define F^i_λ similarly.

Also, denote by $\bar{\lambda}$ the partition λ with the first part removed. We consider generating functions of the form $G^i_{\bar{\lambda}}$. Let λ have b parts, for b fixed. For any $A \subseteq [b]$ and any $1 \le k \le b$, we define $f^\lambda_A(k)$ and $g_A(k)$ as follows: if $k \ge \min(A)$, define $f^\lambda_A(k) = \lambda_{b+1-\max(A\cap[k])} + 1$. Otherwise, define $f^\lambda_A(k) = 0$.

Define $g_A(k) = k - \max(A \cap [k]) + 1$. For convenience, here and in the formula below, we define $\max(\emptyset) = 1$.

Proposition 1.

$$G^i_\lambda(q) = \sum_{A\subseteq[b]} G^i_{\lambda,A}(q) \tag{1}$$

where

$$G^i_{\lambda,A}(q) = (-1)^{|A|} q^{\left(\sum_{k=1}^{b-1} f^\lambda_A(k)\right) + i f^\lambda_A(b)} \prod_{k=1}^{\max(A)-1} \frac{1}{1-q^{g_A(k)}} \cdot \prod_{k=\max(A)}^{b} \frac{1}{1-q^{g_A(k)+i-1}}.$$

The idea for the inductive step in the proof below is based on Lemma 1 from [6] (where it is only used once).

Proof. We induct on b. For $b = 1$,

$$G^i_\lambda(q) = 1 + q^i + \cdots + q^{i\lambda_1} = \frac{1-q^{i(\lambda_1+1)}}{1-q^i} = \frac{1}{1-q^i} - \frac{q^{i(\lambda_1+1)}}{1-q^i},$$

which matches (1).

Assume $b > 1$. If $\mu \subseteq \lambda$ with $\mu_1 < \lambda_1$, then we let $\phi(\mu) = (\mu_1 + 1, \mu_2, \ldots, \mu_b)$. If $\nu \subseteq \lambda$ with $\nu_1 > \nu_2$, we let $\phi^{-1}(\nu) = (\nu_1 - 1, \nu_2, \ldots, \nu_b)$. For any n, clearly ϕ gives a bijection between partitions $\mu \subseteq \lambda$ with $|\mu| = n-1$ and $\mu_1 < \lambda_1$ and partitions $\nu \subseteq \lambda$ with $|\nu| = n$ and $\nu_1 = \nu_2$. Thus taking $(1-q^i)G^i_\lambda(q)$ gives a lot of cancellation. Specifically, we get

$$(1-q^i)G^i_\lambda(q) = G^{i+1}_{\bar{\lambda}}(q) - q^i(q^{i\lambda_1})G^1_{\bar{\lambda}}(q) \tag{2}$$

$$G^i_\lambda(q) = \frac{1}{(1-q^i)}G^{i+1}_{\bar{\lambda}}(q) - \frac{1}{(1-q^i)}(q^{i(\lambda_1+1)})G^1_{\bar{\lambda}}(q) \tag{3}$$

where the first term on the right side of (2) corresponds to ν with $\nu_1 = \nu_2$, while the second term corresponds to μ with $\mu_1 = \lambda_1$.

It is not difficult (with care) to verify that for $A \subseteq [b-1]$,

$$G^i_{\lambda,A}(q) = \frac{1}{(1-q^i)}G^{i+1}_{\bar{\lambda},A}(q)$$

and

$$G^i_{\lambda,A\cup\{b\}}(q) = -\frac{q^{i(\lambda_1+1)}}{(1-q^i)}G^1_{\bar{\lambda},A}(q),$$

so by (3), we get

$$\begin{aligned} G^i_\lambda(q) &= \frac{1}{(1-q^i)}G^{i+1}_{\bar{\lambda}}(q) - \frac{1}{(1-q^i)}(q^{i(\lambda_1+1)})G^1_{\bar{\lambda}}(q) \\ &= \sum_{A\subseteq[b-1]} \frac{1}{(1-q^i)}G^{i+1}_{\bar{\lambda},A}(q) - \sum_{A\subseteq[b-1]} \frac{q^{i(\lambda_1+1)}}{(1-q^i)}G^1_{\bar{\lambda},A}(q) \\ &= \sum_{B\subseteq[b]} G^i_{\lambda,B}(q). \quad \square \end{aligned}$$

We can substitute $i = 1$ (which is the case we are really interested in).