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Note

# Independence and matching number in graphs with maximum degree 4

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ABSTRACT

We prove that $\frac{7}{4}\alpha(G) + \beta(G) \geq n(G)$ and $\alpha(G) + \frac{3}{2}\beta(G) \geq n(G)$ for every triangle-free graph $G$ with maximum degree at most 4, where $\alpha(G)$ is the independence number and $\beta(G)$ is the matching number of $G$, respectively. These results are sharp for a graph on 13 vertices. Furthermore we show $\chi(G) \leq \frac{7}{4}\omega(G)$ for $\{3K_1, K_1 \cup K_5\}$-free graphs, where $\chi(G)$ is the chromatic number and $\omega(G)$ is the clique number of $G$, respectively.
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## 1. Introduction

Inspired by a result of Henning, Löwenstein, and Rautenbach [5] we investigate linear combinations of the independence number and the matching number of graphs.

The intuition is that the matching number and the independence number are negatively correlated, that is, if the matching number is small, then the independence number is large and vice versa. Thus the sum of both is bounded from below.

The investigation of linear combinations of these two parameters one might date back to the famous result of König–Evergáry on maximum matchings in a bipartite graph which says (in an equivalent way) that in a graph $G$ the sum of the matching number and the independence number equals the order of $G$.

We start with some notation. We only consider simple, finite, and undirected graphs. For a graph $G$, we denote by $V(G)$ and $E(G)$ the vertex and edge set, respectively. Let $n(G) = |V(G)|$ be the order and $m(G) = |E(G)|$ be the size of $G$. For a vertex $v \in V(G)$, let $N_G(v)$ be the set of all neighbours of $v$. Define the degree $d_G(v)$ of $v$ by $d_G(v) = |N_G(v)|$. Furthermore, let the closed neighbourhood $N_G[v]$ of $v$ be defined by $N_G[v] = N_G(v) \cup \{v\}$. The maximum degree $\Delta(G)$ of $G$ is the maximum over all vertex degrees in $G$. The minimum degree $\delta(G)$ of $G$ is the minimum over all vertex degrees in $G$. For a set $X \subseteq V(G)$, let $G[X]$ be the subgraph induced by $X$. We write $G \setminus X$ for $G[V(G) \setminus X]$. If $v \in V(G)$ and $e \in E(G)$, we write $G - v$ for $G[V(G) \setminus \{v\}]$ and $G - e$ for the graph in which $e$ is deleted from $G$. We call a set $I$ of vertices of $G$ an independent set of $G$, if no two vertices in $I$ are adjacent. The maximum order of an independent set of $G$ is the independence number $\alpha(G)$. A set $M$ of edges of $G$ is a matching, if no two edges in $M$ are adjacent. The maximum size of a matching in $G$ is the matching number $\beta(G)$. We say that $G$ has a perfect matching, if $\beta(G) = \frac{1}{2}n(G)$. We call a graph $G$ factor-critical, if for every vertex $v \in V(G)$, the graph $G - v$ has a perfect matching. A bridge is an edge $e \in E(G)$ such that $G - e$ has more components than $G$. We call a component $C$ of a graph odd (even), if $n(C)$ is odd (even). Let $o(G)$ be the number of odd components of $G$. For a graph $G$, we call a set $X$ of vertices of $G$ a clique, if all vertices in $X$ are pairwise adjacent. The clique number $\omega(G)$ is the largest order of a clique in $G$. The chromatic number $\chi(G)$ is the smallest integer $k$ such that $V(G)$ has a partition into $k$ independent sets. Now we come to our main results.

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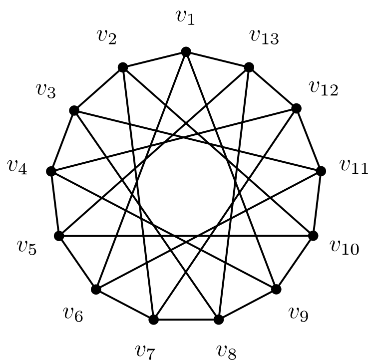


**Fig. 1.** $G_{13}$.

**Theorem 1.** *If $G$ is a triangle-free graph with $\Delta(G) \leq 4$, then*

$$\frac{7}{4}\alpha(G) + \beta(G) \geq n(G)$$

*with equality if and only if every component $C$ of $G$ has order 13, $\alpha(C) = 4$, and $\beta(C) = 6$.*

**Theorem 2.** *If $G$ is a triangle-free graph with $\Delta(G) \leq 4$, then*

$$\alpha(G) + \frac{3}{2}\beta(G) \geq n(G)$$

*with equality if and only if every component $C$ of $G$ is either a single vertex, or a cycle on 5 vertices, or has order 13, $\alpha(C) = 4$ and $\beta(C) = 6$.*

Theorem 2 strengthens a result of Henning, Löwenstein and Rautenbach [5] by relaxing the maximum degree condition from 3 to 4. Theorem 1 leads to some consequences for $\chi$-binding functions. We say that a class $\mathcal{G}$ of graphs has a $\chi$-binding function if there is a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$. The concept of $\chi$-binding functions has a long history and goes back to the work of Gyárfás [4] and Wagon [7]. As mentioned in [5] the class of graphs that satisfy $\alpha(G) \leq 2$ has the $\chi$-binding function $\chi(G) = O\left(\frac{\omega(G)^2}{\log \omega(G)}\right)$.

Choudum et al. [1] give $\chi$-binding functions for graphs with independence number at most 2 and an excluded graph of order 5. Our main result implies the following.

**Corollary 3.** *If $G$ is a graph that contains neither $3K_1$ nor $K_1 \cup K_5$ as an induced subgraph, then*

$$\chi(G) \leq \frac{7}{4}\omega(G).$$

**Proof.** Let $G$ be a graph that neither contains $3K_1$ nor $K_1 \cup K_5$ as an induced subgraph. The complement $\bar{G}$ of $G$ is triangle-free and satisfies $\Delta(\bar{G}) \leq 4$. Since there is no independent set of order at least 3, we obtain $\chi(G) = n(G) - \beta(\bar{G})$. By Theorem 1, we have $\frac{7}{4}\alpha(\bar{G}) + \beta(\bar{G}) \geq n(\bar{G}) = n(G)$ and hence

$$\chi(G) \leq \frac{7}{4}\alpha(\bar{G}) = \frac{7}{4}\omega(G). \quad \Box$$

Our results are best possible. Consider the graph $G_{13}$ on 13 vertices $v_1, \ldots, v_{13}$. Two vertices $v_i$ and $v_j$ such that $i < j$ are joined by an edge if and only if $j - i \in \{1, 5, 8, 12\}$. That means $G_{13}$ is a cycle with all 5-chords. See Fig. 1 for an illustration. Trivially $\beta(G_{13}) = 6$ and to convince ourselves that $\alpha(G_{13}) = 4$, we consider an independent set $I$ of size $\alpha(G_{13})$. By symmetry we assume $v_1 \in I$. Let $G' = G_{13} \setminus N_{G_{13}}[v_1]$. The remaining graph $G'$ is a cycle on 8 vertices $v_3, v_4, v_5, v_{10}, v_{11}, v_{12}, v_7, v_8$ with two 4-chords $v_3v_{11}$ and $v_4v_{12}$. For contradiction we suppose $\alpha(G') \geq 4$ and let $I'$ be an independent set of $G'$ of size 4. Since there is a spanning 8-cycle in $G'$, every second vertex of the cycle is in $I'$; thus either $v_3$ and $v_{11}$ or $v_4$ and $v_{12}$ is in $I'$, which is a contradiction. It follows $\alpha(G') = 3$, because $\{v_3, v_5, v_7\}$ is independent in $G'$. Since $\alpha(G_{13}) = 1 + \alpha(G')$, by symmetry, we conclude $\alpha(G_{13}) = 4$. Thus $\frac{7}{4}\alpha(G_{13}) + \beta(G_{13}) = 13 = n(G_{13})$.
Before we prove Theorems 1 and 2 in Sections 2 and 3, we recall a few results for later use. The first result due to Jones [6].

**Theorem 4** (*Jones [6]*). *If $G$ is a triangle-free graph with $\Delta(G) \leq 4$, then*

$$\alpha(G) \geq \frac{4}{13}n(G).$$

This result leads to a simple consequence.

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