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A characterization of hypergraphs that achieve equality in the Chvátal–McDiarmid Theorem

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ABSTRACT

For $k \geq 2$, let H be a k-uniform hypergraph on n vertices and m edges. The transversal number $\tau(H)$ of H is the minimum number of vertices that intersect every edge. Chvátal and McDiarmid (1992) proved that $\tau(H) \leq (n + \lfloor \frac{k}{2} \rfloor m)/(\lfloor \frac{3k}{2} \rfloor)$. When $k = 3$, the connected hypergraphs that achieve equality in the Chvátal–McDiarmid Theorem were characterized by Henning and Yeo (2008). In this paper, we characterize the connected hypergraphs that achieve equality in the Chvátal–McDiarmid Theorem for $k = 2$ and for all $k \geq 4$.

1. Introduction

In this paper we continue the study of transversals in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, E)$ is a finite set $V = V(H)$ of elements, called *vertices*, together with a finite multiset $E = E(H)$ of subsets of V, called *hyperedges* or simply *edges*.

A *k-edge* in H is an edge of size k. The hypergraph H is said to be *k-uniform* if every edge of H is a k-edge. Every (simple) graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs. The *degree* of a vertex v in H, denoted by $d_H(v)$ or simply by $d(v)$ if H is clear from the context, is the number of edges of H which contain v. The minimum and maximum degrees among the vertices of H are denoted by $\delta(H)$ and $\Delta(H)$, respectively.

Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq e$. The *neighborhood* of a vertex v in H, denoted $N_H(v)$ or simply $N(v)$ if H is clear from the context, is the set of all vertices different from v that are adjacent to v. Two vertices x and y of H are *connected* if there is a sequence $x = v_0, v_1, v_2, \dots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i = 1, 2, \dots, k$. A *connected hypergraph* is a hypergraph in which every pair of vertices are connected. A maximal connected subhypergraph of H is a *component* of H. Thus, no edge in H contains vertices from different components.

If H denotes a hypergraph and X denotes a subset of vertices in H, then $H - X$ will denote that hypergraph obtained from H by removing the vertices X from H, removing all hyperedges that intersect X and removing all resulting isolated vertices, if any. If $X = \{x\}$, we simply denote $H - X$ by $H - x$. We remark that in the literature this is sometimes denoted by *strongly deleting* the vertices in X.

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. A transversal of size $\tau(H)$ is called a $\tau(H)$-set. Transversals in hypergraphs are well studied in the literature (see, for

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0012-365X/

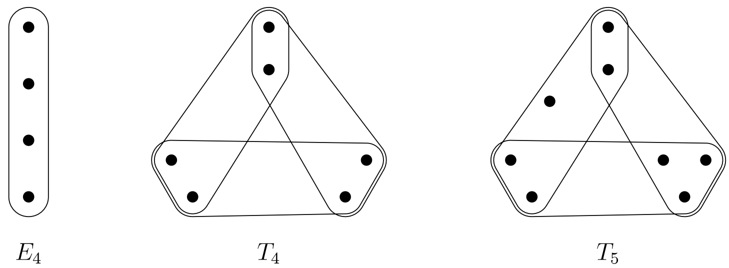

Fig. 1. The hypergraphs E_4, T_4, and T_5.

example, [1,3–6,8,10]). Chvátal and McDiarmid [1] established the following upper bound on the transversal number of a uniform hypergraphs in terms of its order and size.

Chvátal–McDiarmid Theorem. *For $k \ge 2$, if H is a k-uniform hypergraph on n vertices with m edges, then*

$$\tau(H) \le \frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor}.$$

As a special case of the Chvátal–McDiarmid Theorem when $k = 3$, we have that if H is a 3-uniform hypergraph on n vertices with m edges, then $\tau(H) \le (n + m)/4$. This bound was independently established by Tuza [11] and a short proof of this result was also given by Thomassé and Yeo [10]. The extremal connected hypergraphs that achieve equality in the Chvátal–McDiarmid Theorem when $k = 3$ were characterized by Henning and Yeo [4]. Their characterization showed that there are three infinite families of extremal connected hypergraphs, as well as two special hypergraphs, one of order 7 and the other of order 8.

Our aim in this paper is to characterize the connected hypergraphs that achieve equality in the Chvátal–McDiarmid Theorem for $k = 2$ and for all $k \ge 4$. For this purpose we define two special families of hypergraphs.

1.1. Special families of hypergraphs

For $k \ge 2$, let E_k denote the k-uniform hypergraph on k vertices with exactly one edge. The hypergraph E_4 is illustrated in Fig. 1.

For $k \ge 2$, a *generalized triangle* T_k is defined as follows. Let A, B, C and D be vertex-disjoint sets of vertices with $|A| = \lceil k/2 \rceil$, $|B| = |C| = \lfloor k/2 \rfloor$ and $|D| = \lceil k/2 \rceil - \lfloor k/2 \rfloor$. In particular, if k is even, the set $D = \emptyset$, while if k is odd, the set D consist of a singleton vertex. Let T_k denote the k-uniform hypergraph with $V(T_k) = A \cup B \cup C \cup D$ and with $E(T_k) = \{e_1, e_2, e_3\}$, where $V(e_1) = A \cup B$, $V(e_2) = A \cup C$, and $V(e_3) = B \cup C \cup D$. The hypergraphs T_4 and T_5 are illustrated in Fig. 1.

We shall need the following properties of special hypergraphs defined in this section.

Observation 1. Let $k \ge 2$ and let $H = E_k$ or $H = T_k$ and let H have n vertices and m edges. Then the following holds.

(a) If $H = E_k$, then $\tau(H) = 1$.
(b) If $H = T_k$, then $\tau(H) = 2$.
(c) $\tau(H) = (n + \lfloor \frac{k}{2} \rfloor m)/\lfloor \frac{3k}{2} \rfloor$.
(d) Every vertex in H belongs to some $\tau(H)$-set.

2. Main result

We shall prove:

Theorem 2. *For $k = 2$ or $k \ge 4$, let H be a connected k-uniform hypergraph on n vertices and m edges. Then,*

$$\tau(H) \le \frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor}$$

with equality if and only if $H = E_k$ or $H = T_k$.

We remark that if we relax the connectivity condition in the statement of Theorem 2, then equality occurs if and only if every component of H is either E_k or T_k. We proceed as follows. We first recall some important results on edge colorings of multigraphs in Section 3. Thereafter we establish a key theorem about matchings in multigraphs in Section 4. Finally in Section 5 we present a proof of Theorem 2 using an interplay between transversals in hypergraphs and matchings in multigraphs.