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The Ramsey numbers of wheels versus odd cycles

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ABSTRACT

Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that for any graph G of order N, either G contains G_1 or its complement contains G_2. Let C_m denote a cycle of order m and W_n a wheel of order $n + 1$. In this paper, it is shown that $R(W_n, C_m) = 2n + 1$ for m odd, $n \geq 3(m - 1)/2$ and $(m, n) \neq (3, 3), (3, 4)$, and $R(W_n, C_m) = 3m - 2$ for m, n odd and $m < n \leq 3(m - 1)/2$.

1. Introduction

We are here concerned with finite simple graphs. Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. Let $G = (V(G), E(G))$ and $e(G) = |E(G)|$. For $S \subseteq V(G)$, let $G[S]$ and $G - S$ denote the subgraph induced by S and $V(G) - S$, respectively. Moreover, $N_S(v)$ and $d_S(v)$ are the set and the number of the neighbors of a vertex v contained in S, respectively. If $S = V(G)$, we write $N(v) = N_G(v)$, $N[v] = N(v) \cup \{v\}$ and $d(v) = d_G(v)$. For $V_1, V_2 \subseteq V(G)$, we use $V_1 - V_2$ to denote the vertices contained in V_1 but not in V_2, $E(V_1, V_2)$ the set of edges between V_1 and V_2 and $e(V_1, V_2) = |E(V_1, V_2)|$. A cycle and a path of order m are denoted by C_m and P_m, respectively. An (x, y)-path is a path from x to y. We denote by $\overrightarrow{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$ then $u\overrightarrow{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. We use K_n to denote a complete graph of order n and $K_{l_1,l_2,\ldots,l_k}$ a complete k-partite graph of order $\sum_{i=1}^{k} l_i$. A wheel $W_n = K_1 + C_n$ is a graph of order $n + 1$. The minimum degree, maximum degree and independence number of G are denoted by $\delta(G)$, $\Delta(G)$ and $\alpha(G)$, respectively. We use mG to denote m vertex disjoint copies of G. The lengths of the longest and shortest cycles of G are denoted by $c(G)$ and $g(G)$, respectively. A graph G of order n is called Hamiltonian, pancyclic and weakly pancyclic if it contains C_n, cycles of every length between 3 and n and cycles of every length l with $g(G) \leq l \leq c(G)$, respectively.

In [3], Burr defined a connected graph F to be H-good, if

$$R(F, H) = (|F| - 1)(\chi(H) - 1) + s(H) \quad \text{for } |F| \geq s(H),$$

where $\chi(H)$ denotes the chromatic number of H and $s(H)$ the chromatic surplus of H, i.e., the minimum number of vertices in some color class under all proper vertex colorings by $\chi(H)$ colors.

For the cycle–wheel Ramsey number, Burr and Erdős first proved that W_n is C_3-good when $n \geq 5$, which was the first paper involving the pair of a cycle and a wheel.

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Theorem 1 (*Burr and Erdős [4]*). *$R(W_n, C_3) = 2n + 1$ for $n \geq 5$.*

Surahmat et al. [13,14] conjectured that C_m is W_n-good if $m \geq n$ and $(m, n) \neq (3, 3), (4, 4)$ and got the following result.

Theorem 2 (*Surahmat et al. [13,14]*). *$R(C_m, W_n) = 2m - 1$ for even n and $m \geq 5n/2 - 1$, and $R(C_m, W_n) = 3m - 2$ for odd $n \geq 5$ and $m > (5n - 9)/2$.*

Chen et al. [5] showed that the range in Theorem 2 can be improved to $m \geq 3n/2 + 1$ for even $n \geq 4$ and established the following.

Theorem 3 (*Chen et al. [5]*). *$R(C_m, W_n) = 2m - 1$ even $n \geq 4$ and $m \geq 3n/2 + 1$.*

By discussing the relations between the size and the weakly pancyclic property in a graph and its complement, Chen et al. [6] also determined that C_m is W_n-good for odd n, $m \geq n \geq 3$ and $(m, n) \neq (3, 3)$.

Theorem 4 (*Chen et al. [6]*). *$R(C_m, W_n) = 3m - 2$ for odd n, $m \geq n \geq 3$ and $(m, n) \neq (3, 3)$.*

In the case when $m \leq n - 1$, Zhou [16] showed that W_n is C_m-good if m is odd and $n \geq 5m - 7$. Unfortunately, the correctness of the proof is questionable since the author did not give the proofs for the two key claims in the paper. Recently, Sun and Chen [12] considered the wheels which are C_5-good and obtained the following.

Theorem 5 (*Sun and Chen [12]*). *$R(W_n, C_5) = 2n + 1$ for $n \geq 6$.*

Up to now, whether W_n is C_m-good for odd $m > 5$ is still open. Other results on Ramsey numbers of cycles versus wheels can be found in the dynamic survey [11]. In this paper, we determine the values of $R(W_n, C_m)$ for odd m in a more general situation. The main results of this paper are as follows.

Theorem 6. *$R(W_n, C_m) = 2n + 1$ for m odd, $n \geq 3(m - 1)/2$ and $(m, n) \neq (3, 3), (3, 4)$.*

Theorem 7. *$R(W_n, C_m) = 3m - 2$ for m, n odd and $m < n \leq 3(m - 1)/2$.*

Clearly, Theorem 6 says that W_n is C_m-good for odd $m \geq 3$, $n \geq 3(m - 1)/2$ and $(m, n) \neq (3, 3), (3, 4)$, and Theorem 7 shows that C_m is W_n-good for odd m, n and $m < n \leq 3(m - 1)/2$.

2. Preliminary lemmas

In order to prove Theorems 6 and 7, we need the following lemmas.

Lemma 1 (*Brandt [1]*). *Every nonbipartite graph G of order n with $e(G) > (n - 1)^2/4 + 1$ is weakly pancyclic with $g(G) = 3$.*

Lemma 2 (*Brandt et al. [2]*). *Every nonbipartite graph G of order n with $\delta(G) \geq (n+2)/3$ is weakly pancyclic with $g(G) = 3$ or 4.*

Lemma 3 (*Dirac [7]*). *Let G be a simple graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian.*

Lemma 4 (*Dirac [7]*). *Let G be a graph with $\delta(G) \geq 2$, then $c(G) \geq \delta(G) + 1$.*

Lemma 5 (*Erdős and Gallai [8]*). *Let G be a graph of order n and $3 \leq c \leq n$. If $e(G) \geq (c - 1)(n - 1)/2 + 1$, then $c(G) \geq c$.*

Lemma 6 (*Faudree et al. [9]*). *Let G be a graph of order $n \geq 6$. Then $\max\{c(G), c(\overline{G})\} \geq \lceil 2n/3 \rceil$.*

Lemma 7 (*Lawrence [10]*). *$R(C_m, K_{1,n}) = m$ for $m \geq 2n$.*

Lemma 8. *Let C be a longest cycle of a graph G and $v_1, v_2 \in V(G) - V(C)$. Then $|N_C(v_1) \cup N_C(v_2)| \leq \lfloor |C|/2 \rfloor + 1$.*

Proof. Let $C = u_1u_2 \cdots u_lu_1$. If there exist $u_iu_{i+1}, u_ju_{j+1}, u_ku_{k+1} \in E(C)$ with $i < j < k$ such that $u_i, u_{i+1}, u_j, u_{j+1}, u_k, u_{k+1} \in N_C(v_1) \cup N_C(v_2)$, where the subscripts are taken modulo l, then v_1 or v_2 has at least two neighbors in $\{u_i, u_j, u_k\}$. By symmetry, we may assume that $u_i, u_j \in N_C(v_1)$. By the maximality of C, $u_{i+1}, u_{j+1} \notin N_C(v_1)$ which implies that $u_{i+1}, u_{j+1} \in N_C(v_2)$. Thus, $v_1u_j \overleftarrow{C} u_{i+1}v_2u_{j+1} \overrightarrow{C} u_iv_1$ is cycle of length longer than C, a contradiction. Therefore, C has at most two edges whose ends are contained in $N_C(v_1) \cup N_C(v_2)$, and hence $|N_C(v_1) \cup N_C(v_2)| \leq \lfloor |C|/2 \rfloor + 1$. ■

Lemma 9. *Let $m \geq 5$ be an odd integer and (X, Y) a partition of $V(G)$ of a graph G such that $|Y| \geq (m + 1)/2$ and $|X - (N(y_i) \cup N(y_j))| \geq (m - 1)/2$ for any $y_i, y_j \in Y$. If $\overline{G}$ contains no C_m, then $G[Y]$ is a complete graph.*

Proof. Set $Y = \{y_1, y_2, \ldots, y_k\}$ and $l = (m + 1)/2$, then $k \geq l$. If $G[Y]$ is not a complete graph, say, $y_1y_2 \notin E(G)$, then since $|X - (N(y_i) \cup N(y_j))| \geq (m - 1)/2$ for any $y_i, y_j \in Y$, we can choose $x_1, x_2, \ldots, x_{l-1} \in X$ such that $y_i, y_{i+1} \notin N(x_{i-1})$ for $i = 2, \ldots, l - 1$ and $y_1, y_l \notin N(x_{l-1})$, which implies that $y_1y_2x_1y_3x_2y_4 \cdots x_{l-2}y_lx_{l-1}y_1$ is C_m in $\overline{G}$, a contradiction. ■