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Domination game: Effect of edge- and vertex-removal

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ABSTRACT

The domination game is played on a graph G by two players, named Dominator and Staller. They alternatively select vertices of G such that each chosen vertex enlarges the set of vertices dominated before the move on it. Dominator's goal is that the game is finished as soon as possible, while Staller wants the game to last as long as possible. It is assumed that both play optimally. Game 1 and Game 2 are variants of the game in which Dominator and Staller has the first move, respectively. The game domination number $\gamma_g(G)$ and the Staller-start game domination number $\gamma'(G)$ are the number of vertices chosen in Game 1 and Game 2, respectively. It is proved that if $e \in E(G)$, then $|\gamma_g(G) - \gamma_g(G-e)| \le 2$ and $|\gamma'(G) - \gamma'(G-e)| \le 2$, and that each of the possibilities here is realizable by connected graphs G for all values of $\gamma_g(G)$ and $\gamma'(G)$ larger than 5. For the remaining small values it is either proved that realizations are not possible or realizing examples are provided. It is also proved that if $v \in V(G)$, then $\gamma_g(G) - \gamma_g(G-v) \le 2$ and $\gamma'(G) - \gamma'(G-v) \le 2$. Possibilities here are again realizable by connected graphs G in almost all the cases, the exceptional values are treated similarly as in the edge-removal case.

1. Introduction

The domination game is played on an arbitrary graph G by two players, *Dominator* and *Staller*. They are taking turns choosing a vertex from G such that whenever they choose a vertex, it dominates at least one previously undominated vertex. The game ends when all vertices of G are dominated, so that the set of vertices selected at the end of the game is a dominating set of G. The aim of Dominator (Staller) is that the total number of moves played in the game is as small (as large, resp.) as possible. By *Game* 1 (*Game* 2) we mean a game in which Dominator (Staller, resp.) has the first move. Assuming that both players play optimally, the *game domination number* $\gamma_g(G)$ (the *Staller-start game domination number* $\gamma_g'(G)$) of a graph G, denotes the number of vertices chosen in Game 1 (Game 2, resp.).

Note that the domination game is not a combinatorial game in the strict sense of [5], where the outcome of a game is assumed to be only of the types (lose, win), (tie, tie) and (draw, draw) for the two players.

The domination game was introduced in [2] (with the idea going back to [7]) and explored by now from several points of view. Despite the fact that $\gamma(G) \le \gamma_g(G) \le 2\gamma(G) - 1$ holds for any graph G (see [2]), the game domination number

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is essentially different from the domination number. First of all, $\gamma_g(G)$ is generally much more difficult to determine than $\gamma(G)$. Even on simple graphs such as paths and cycles, the problem of determining γ_g is non-trivial [8].

As proved in [2,9], the game domination number and the Staller-start game domination number can differ only by 1: $|\gamma_g(G) - \gamma_g'(G)| \le 1$. Call a pair of integers (k, ℓ) *realizable* if there exists a graph G with $\gamma_g(G) = k$ and $\gamma_g'(G) = \ell$. Some classes of graphs for possible realizable pairs are given in [2,3,12]. For the complete answer that all pairs that are potentially realizable can be realized (with relatively simple families of graphs) see [10].

Kinnersley, West, and Zamani [9] conjectured that if G is an isolate-free forest of order n or an isolate-free graph of order n, then $\gamma_g(G) \le 3n/5$. Actually they posed two conjectures, because while the truth for isolate-free graphs clearly implies the truth for isolate-free forests, it is not known whether the converse implication holds. These conjectures are known as the 3/5-conjectures. In [1] large families of trees were constructed that attain the conjectured 3/5-bound and all extremal trees on up to 20 vertices were found; in particular, there are exactly ten trees T on 20 vertices with $\gamma_g(T) = 12$. Further progress on the 3/5-conjecture for forest was very recently done by Bujtás [4] by proving that the 3/5-conjecture holds for the class of forests in which no two leaves are at distance 4.

Clearly, removing an edge from a graph cannot decrease its domination number, that is, $\gamma(G - e) \ge \gamma(G)$. (For an extensive survey on graphs that are domination critical with respect to edge- and vertex-removal see [11].) On the other hand, it was proved in [3] that for any integer $\ell \ge 1$, there exist a graph G and a spanning tree T such that $\gamma_g(T) \le \gamma_g(G) - \ell$. In this paper we answer the question how much $\gamma_g(G)$ and $\gamma_g'(G)$ can change if an edge is removed from G. The answer is given in Theorem 2.1 which is followed by ten subsections in which each of the possibilities indicated by the theorem, is shown to be realizable by connected graphs. We also ask the analogous question for vertex-removal and present the answer in Theorem 3.1. Again, all possibilities can be realized by connected graphs. We conclude the paper with some natural open problems, concerning extensions or generalizations of the results from this paper.

For a vertex subset S of a graph G, let $G|S$ denote the graph G in which vertices from S are considered as being already dominated. In particular, if $S = \{x\}$ we write $G|x$. When describing a strategy for a given player, we often use the phrase that the player *follows* the other player in some (sub)graph. This means that if the last move of the other player was on a (sub)graph G, then the next move of the player is a vertex of G. In the case that no legal move exists on G, the next move of the player will be further defined. For all the other standard notions not defined in this paper see the monograph on graph domination [6].

In the rest of this section we state some known results to be used in the sequel.

Theorem 1.1 (*[9, Lemma 2.1]—Continuation Principle*). *Let G be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \le \gamma_g(G|B)$ and $\gamma_g'(G|A) \le \gamma_g'(G|B)$.*

Theorem 1.2 (*[2,9]*). *If G is any graph, then $|\gamma_g(G) - \gamma_g'(G)| \le 1$.*

Theorem 1.3 (*[8]*). *If $n \ge 3$, then*

$$\gamma_g(C_n) = \gamma_g(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 3 \bmod 4, \\ \left\lceil \frac{n}{2} \right\rceil; & \textit{otherwise}. \end{cases}$$

$$\gamma_g'(P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

$$\gamma_g'(C_n) = \begin{cases} \left\lceil \frac{n-1}{2} \right\rceil - 1; & n \equiv 2 \bmod 4, \\ \left\lceil \frac{n-1}{2} \right\rceil; & \textit{otherwise}. \end{cases}$$

Theorem 1.4 (*[9, Theorem 4.6]*). *If F is a forest and $S \subseteq V(F)$, then $\gamma_g(F|S) \le \gamma_g'(F|S)$.*

2. Edge removal

Theorem 2.1. *If G is a graph and $e \in E(G)$, then*

$$|\gamma_g(G) - \gamma_g(G - e)| \le 2 \quad \textit{and} \quad |\gamma_g'(G) - \gamma_g'(G - e)| \le 2.$$

Proof. To prove the bound $\gamma_g(G - e) \le \gamma_g(G) + 2$ it suffices to show that Dominator has a strategy on $G - e$ such that at most $\gamma_g(G) + 2$ moves are played. His strategy is to play the game on $G - e$ as follows. In parallel to the real game he is playing an *imagined game* on G by copying every move of Staller to this game and responding optimally in G. Each response in the imagined game is then copied back to the real game in $G - e$. Let $e = uv$ and consider the following possibilities.

Suppose first that neither Staller nor Dominator play on either of u and v in the course of the real game. This makes all the moves in both games legal and so the imagined game on G lasts no more than $\gamma_g(G)$ moves. (Recall that Dominator plays