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Chiral extensions of chiral polytopes

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ABSTRACT

Given a chiral d-polytope $\mathcal{K}$ with regular facets, we describe a construction for a chiral $(d+1)$-polytope $\mathcal{P}$ with facets isomorphic to $\mathcal{K}$. Furthermore, $\mathcal{P}$ is finite whenever $\mathcal{K}$ is finite. We provide explicit examples of chiral 4-polytopes constructed in this way from chiral toroidal maps.

1. Introduction

Abstract polytopes are combinatorial structures that mimic convex polytopes in several key ways. They also generalize (non-degenerate) maps on surfaces and face-to-face tessellations of euclidean, hyperbolic, and projective spaces. Regular polytopes have full symmetry by (abstract) reflections and have been extensively studied [7]. One of the most important problems in the study of regular polytopes is the *extension problem*: given a regular polytope $\mathcal{K}$ of rank d, what sorts of regular polytopes of rank $d+1$ have facets isomorphic to $\mathcal{K}$? Though this problem is far from solved, many useful partial results already exist (see, for example, [4,14]). In particular, if $\mathcal{K}$ is a finite regular polytope, then [10] shows how to construct infinitely many finite regular polytopes with facets isomorphic to $\mathcal{K}$.

Another important class of polytopes are the *chiral polytopes*, which have full rotational symmetry, but no symmetry by reflection. There are many examples in ranks 3 and 4 (see [1,16] for some of these). In higher ranks, however, we have only a handful of concrete examples.

Many of the important unsolved problems of chiral polytopes are summarized in [12]. Problems 24–30 all concern the extension problem for chiral polytopes, signifying both the importance of that general problem and how little is known. An important partial result was given in [11], where it is shown how to build a finite chiral polytope of rank $d+1$ with facets isomorphic to a finite regular polytope $\mathcal{K}$ of rank d. There are very restrictive conditions on the polytope $\mathcal{K}$, however, so more work remains to be done even on this piece of the extension problem (Problem 27 of [12]).

In this paper we use GPR graphs (as defined in [13]) to build chiral polytopes of rank $d+1$ with facets isomorphic to a given chiral polytope of rank d. In particular, Theorem 10 implies the following:

Theorem 1. *Every finite chiral d-polytope with regular facets is itself the facet of a finite chiral $(d+1)$-polytope.*

This gives a partial answer to Problem 26 in [12]. We note that the assumption that the chiral d-polytope has regular facets is necessary (see [15, Proposition 9]).

We will start by giving background on polytopes in Section 2 and on GPR graphs in Section 3. Section 4 details the main construction, culminating in Theorem 10. Finally, in Section 5, we will apply the construction to a family of chiral toroidal maps $\{4, 4\}_{(b,c)}$ and analyze the structure of the resulting chiral polytope of rank 4.

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2. Regular and chiral polytopes

In this section we introduce abstract regular and chiral polytopes, referring to [7,15] for details.

An *(abstract) d-polytope* $\mathcal{K}$ is a partially ordered set whose elements are called *faces* and which satisfies the following properties. It contains a minimum face F_{-1} and a maximum face F_d, and every *flag* of $\mathcal{K}$ (maximal totally ordered subset) contains precisely $d+2$ elements, including F_{-1} and F_d. This induces a rank function from $\mathcal{K}$ to the set $\{-1, 0, \ldots, d\}$ such that $rank(F_{-1}) = -1$ and $rank(F_d) = d$. The faces of rank i are called *i-faces*, the 0-faces are called *vertices*, the 1-faces are called *edges* and the $(d-1)$-faces are called *facets*. Furthermore, we say that $\mathcal{K}$ has rank d. In analogy with convex polytopes, an abstract 3-polytope is also called a *polyhedron*. We shall abuse notation and identify the *section* $G/F_{-1} := \{H \mid H \leq G\}$ with the face G itself. Given a vertex v, the section $F_d/v := \{H \mid H \geq v\}$ is called the *vertex-figure* of $\mathcal{K}$ at v. For every pair of incident faces $F \leq G$ such that $rank(G) - rank(F) = 2$, there exist precisely two faces H_1 and H_2 such that $F < H_1, H_2 < G$. This property is referred to as the *diamond condition*. As a consequence of the diamond condition, for any flag Φ and any $i \in \{0, \ldots, d-1\}$ there exists a unique flag Φ^i that differs from Φ only in the i-face. This flag is called the *i-adjacent flag* of Φ. Finally, $\mathcal{K}$ must be *strongly flag-connected*, meaning that for any two flags Φ, Φ' there exists a sequence of flags $\Phi = \Psi_0, \Psi_1, \ldots, \Psi_m = \Phi'$ such that $\Phi \cap \Phi' \subseteq \Psi_k$, and Ψ_{k-1} is adjacent to Ψ_k for $k = 1, \ldots, m$.

If F is an $(i-2)$-face and G is an $(i+1)$-face of the d-polytope $\mathcal{K}$, with $F < G$, then the section $G/F := \{H \mid F \leq H \leq G\}$ is an abstract polygon. If $\mathcal{K}$ has the property that the type of each of these sections depends only on i (and not on the particular choice of F and G), then we say that $\mathcal{K}$ is *equivelar*. In this case, $\mathcal{K}$ has a *Schläfli type* (or *Schläfli symbol*) $\{p_1, \ldots, p_{d-1}\}$, where the section G/F is a p_i-gon whenever F is an $(i-2)$-face and G is an $(i+1)$-face with $F < G$. The numbers p_i satisfy $2 \leq p_i \leq \infty$, but in this paper we will always have $3 \leq p_i$. Regular and chiral polytopes, defined below, are examples of equivelar polytopes.

An *automorphism* of a d-polytope $\mathcal{K}$ is an order-preserving permutation of its faces. The group of automorphisms of $\mathcal{K}$ is denoted by $\Gamma(\mathcal{K})$. There is a natural action of $\Gamma(\mathcal{K})$ on the flags of $\mathcal{K}$, and we say that $\mathcal{K}$ is *regular* if this action is transitive. In this case, $\Gamma(\mathcal{K})$ is generated by involutions $\rho_0, \ldots, \rho_{d-1}$, where ρ_i is the unique automorphism mapping a fixed *base flag* Φ to its i-adjacent flag Φ^i. These generators satisfy the relations

$$\rho_i^2 = \varepsilon,$$
$$(\rho_i\rho_j)^2 = \varepsilon \quad \text{whenever } |i-j| \geq 2,$$

where ε denotes the identity element. Regular polytopes are equivelar, and the order of the element $\rho_{i-1}\rho_i$ coincides with the Schläfli number p_i.

The generators $\{\rho_0, \ldots, \rho_{d-1}\}$ also satisfy the intersection conditions given by

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle, \tag{1}$$

for all $I, J \subseteq \{0, \ldots, d-1\}$.

A *string C-group* is a group together with a generating set $\{\rho_0, \ldots, \rho_{d-1}\}$ such that the generators ρ_i are involutions satisfying the relation $(\rho_i\rho_j)^2 = \varepsilon$ for $|i-j| \geq 2$, and the intersection condition (1). The string C-groups are in a one-to-one correspondence with the automorphism groups of regular polytopes; in particular, every regular polytope can be reconstructed from its automorphism group.

The *rotation subgroup* of (the automorphism group of) a regular d-polytope $\mathcal{K}$ is defined as the subgroup $\Gamma^+(\mathcal{K})$ of $\Gamma(\mathcal{K})$ consisting of all elements that can be expressed as words of even length on the generators $\rho_0, \ldots, \rho_{d-1}$. The index of $\Gamma^+(\mathcal{K})$ in $\Gamma(\mathcal{K})$ is at most 2. Whenever $\Gamma^+(\mathcal{K})$ has index 2 in $\Gamma(\mathcal{K})$ we say that $\mathcal{K}$ is *orientably regular*; other sources also use the term *directly regular* (see, for example, [15]).

For $i = 1, \ldots, d-1$ we define the *abstract rotation* σ_i to be $\rho_{i-1}\rho_i$, that is, the automorphism of $\mathcal{K}$ mapping the base flag Φ to $(\Phi^i)^{i-1}$. Then $\Gamma^+(\mathcal{K}) = \langle \sigma_1, \ldots, \sigma_{d-1} \rangle$ and the abstract rotations satisfy the relations

$$(\sigma_i \cdots \sigma_j)^2 = \varepsilon \tag{2}$$

for $i < j$. The order of σ_i is just the entry p_i in the Schläfli symbol.

We define *abstract half-turns* as the involutions $\tau_{i,j} := \sigma_i \cdots \sigma_j$ for $i < j$. For consistency we also define $\tau_{0,i} := \tau_{i,d} := \varepsilon$ and denote σ_i by $\tau_{i,i}$. Then the abstract rotations and half-turns satisfy the intersection condition given by

$$\langle \tau_{i,j} \mid i \leq j;\, i-1, j \in I \rangle \cap \langle \tau_{i,j} \mid i \leq j;\, i-1, j \in J \rangle = \langle \tau_{i,j} \mid i \leq j;\, i-1, j \in I \cap J \rangle \tag{3}$$

for $I, J \subseteq \{-1, \ldots, d\}$.

We say that the d-polytope $\mathcal{K}$ is *chiral* if its automorphism group induces two orbits on flags with the property that adjacent flags always belong to different orbits. The facets and vertex-figures of a chiral polytope must be either orientably regular or chiral, and the $(d-2)$-faces must be orientably regular (see [15, Proposition 9]).

The automorphism group $\Gamma(\mathcal{K})$ of a chiral polytope is generated by elements $\sigma_1, \ldots, \sigma_{d-1}$, where σ_i maps a base flag Φ to $(\Phi^i)^{i-1}$. That is, σ_i cyclically permutes the i- and $(i-1)$-faces of $\mathcal{K}$ incident with the $(i-2)$- and $(i+1)$-faces of Φ. Furthermore, the generators σ_i also satisfy (2) as well as the intersection conditions (3). Because of the obvious similarities between the automorphism group of a chiral polytope and the rotation subgroup of a regular polytope we shall also refer to the generators σ_i of the automorphism group of a chiral polytope as *abstract rotations*, to the products