Discrete Mathematics 324 (2014) 13–18

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On matching and semitotal domination in graphs

Michael A. Henning*, Alister J. Marcon

Department of Mathematics, University of Johannesburg, Auckland Park, 2006, South Africa

ARTICLE INFO

Article history:
Received 11 October 2013
Received in revised form 22 January 2014
Accepted 26 January 2014
Available online 10 February 2014

Keywords:
Domination
Total domination
Semitotal domination
Matching

ABSTRACT

Let G be a graph with no isolated vertex. In this paper, we study a parameter that is squeezed between arguably the two most important domination parameters, namely the domination number, $\gamma(G)$, and the total domination number, $\gamma_t(G)$. A set S of vertices in G is a semitotal dominating set of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The semitotal domination number, $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set of G. We observe that $\gamma(G) \le \gamma_{t2}(G) \le \gamma_t(G)$. It is known that $\gamma(G) \le \alpha'(G)$, where $\alpha'(G)$ denotes the matching number of G. However, the total domination number and the matching number of a graph are generally incomparable. We provide a characterization of minimal semitotal dominating sets in graphs. Using this characterization, we prove that if G is a connected graph on at least two vertices, then $\gamma_{t2}(G) \le \alpha'(G) + 1$ and we characterize the graphs achieving equality in the bound.

1. Introduction

A *dominating set* in a graph G is a set S of vertices of G such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [6,7].

A *total dominating set*, abbreviated a TD-set, of a graph G with no isolated vertex is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. Total domination is now well studied in graph theory. The literature on the subject of total domination in graphs has been surveyed and detailed in the recent book [10]. A survey of total domination in graphs can also be found in [8].

In this paper, we continue the study of semitotal domination in graphs introduced and studied by Goddard, Henning and McPillan [5]. A set S of vertices in a graph G with no isolated vertices is a *semitotal dominating set*, abbreviated semi-TD-set, of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The *semitotal domination number*, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semi-TD-set of G. A semi-TD-set of G of cardinality $\gamma_{t2}(G)$ is called a *$\gamma_{t2}(G)$-set*.

Two edges in a graph G are *independent* if they are not adjacent in G. A *matching* in G is a set of (pairwise) independent edges, while a matching of maximum cardinality is a *maximum matching*. The cardinality of a maximum matching is the *matching number* of G which we denote by $\alpha'(G)$. Matchings in graphs are extensively studied in the literature (see, for example, the excellent survey articles by Plummer [15] and Pulleyblank [16]).

Since every TD-set is a semi-TD-set, and since every semi-TD-set is a dominating set, we have the following observation.

Observation 1. *For every graph G with no isolated vertex,* $\gamma(G) \le \gamma_{t2}(G) \le \gamma_t(G)$.

* Corresponding author.
E-mail addresses: mahenning@uj.ac.za (M.A. Henning), alister.marcon@gmail.com (A.J. Marcon).

http://dx.doi.org/10.1016/j.disc.2014.01.021
0012-365X/

By Observation 1, the semitotal domination number is squeezed between arguably the two most important domination parameters, namely the domination number and the total domination number. Our aim in this paper is twofold. Our first aim is to provide a characterization of minimal semi-TD-sets in graphs. Our second aim is to prove that if G is a connected graph on at least two vertices, then $\gamma_{t2}(G) \le \alpha'(G) + 1$ and to characterize the graphs achieving equality in the bound.

1.1. Terminology and notation

For notation and graph theory terminology that are not defined herein, we refer the reader to [10]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $n = |V|$ and edge set $E = E(G)$ of size $m = |E|$, and let v be a vertex in V. We denote the *degree* of v in G by $d_G(v)$. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively). A *leaf* of G is a vertex of degree 1. A *branch vertex* is a vertex of degree at least 3 in G.

For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. A *cycle* and *path* on n vertices are denoted by C_n and P_n, respectively. For two vertices u and v in a connected graph G, the *distance* $d_G(u, v)$ between u and v is the length of a shortest u–v path in G. The distance $d_G(v, S)$ between a vertex v and a set S of vertices in a graph G is the minimum distance from v to a vertex of S in G.

The *open neighborhood* of a vertex v is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. The *open-2-neighborhood* of v is the set $N_2(v, G) = \{u \in V \mid d_G(u, v) = 2\}$ of vertices at distance 2 from v in G. The *closed-2-neighborhood* of v is the set

$$N_{\le 2}[v, G] = N_G[v] \cup N_2(v, G)$$

of all vertices within distance 2 from v, while the *inclusive open-2-neighborhood* of v is the set $N_{\le 2}(v, G) = N_G(v) \cup N_2(v, G)$. For a set $S \subseteq V$, its *open neighborhood* is the set

$$N_G(S) = \bigcup_{v \in S} N_G(v),$$

and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. The *S-private neighborhood* of v in G is defined by

$$\mathrm{pn}_G[v, S] = \{w \in V \mid N_G[w] \cap S = \{v\}\},$$

while its *open S-private neighborhood* is defined by

$$\mathrm{pn}_G(v, S) = \{w \in V \mid N_G(w) \cap S = \{v\}\}.$$

We remark that the sets $\mathrm{pn}_G[v, S] \setminus S$ and $\mathrm{pn}_G(v, S) \setminus S$ are equivalent and define the *S-external private neighborhood* of v to be this set, abbreviated $\mathrm{epn}_G[v, S]$ or $\mathrm{epn}_G(v, S)$. The *S-internal private neighborhood* of v is defined by $\mathrm{ipn}_G[v, S] = \mathrm{pn}_G[v, S] \cap S$ and its *open S-internal private neighborhood* is defined by $\mathrm{ipn}_G(v, S) = \mathrm{pn}_G(v, S) \cap S$. Hence,

$$\mathrm{pn}_G(v, S) = \mathrm{epn}_G(v, S) \cup \mathrm{ipn}_G(v, S).$$

We define an *S-external private neighbor* of v to be a vertex in $\mathrm{epn}_G(v, S)$ and an *S-internal private neighbor* of v to be a vertex in $\mathrm{ipn}_G(v, S)$. We remark that either v is isolated in $G[S]$—that is, $\mathrm{ipn}_G[v, S] = \{v\}$—or v has at least one neighbor in S—in which case, $\mathrm{ipn}_G[v, S] = \emptyset$. Thus, $\mathrm{ipn}_G[v, S] \in \{\emptyset, \{v\}\}$.

The *S-private 2-neighborhood* of v in G is defined by

$$\mathrm{pn}_2[v, S; G] = \{w \in V \mid N_{\le 2}[w, G] \cap S = \{v\}\},$$

while its *open S-private 2-neighborhood* is defined by

$$\mathrm{pn}_2(v, S; G) = \{w \in V \mid N_{\le 2}(w, G) \cap S = \{v\}\}.$$

The *open S-internal private 2-neighborhood* of v is defined by

$$\mathrm{ipn}_2(v, S; G) = \mathrm{pn}_2(v, S; G) \cap S.$$

If the graph G is clear from the context, we omit it in the above expressions. For example, we write $N_{\le 2}[v]$ and $N_{\le 2}(v)$ rather than $N_{\le 2}[v, G]$ and $N_{\le 2}(v, G)$, respectively.

Let X and Y be subsets of vertices in G. If $Y \subseteq N[X]$, then we say the set X *dominates* the set Y in G and that the set Y is *dominated* by X. Furthermore, if $Y = \{y\}$, then we simply say that y is dominated by X rather than $\{y\}$ is dominated by X. Thus, if a vertex v is dominated by X, then $N[v] \cap X \ne \emptyset$. We note that if X dominates V, then X is a dominating set in G. Hence, if X is a dominating set in G, then $N[X] = V$.

2. Minimal semitotal dominating set

A *minimal TD-set* in G is a TD-set that contains no TD-set of G as a proper subset. The following property of a minimal TD-set in a graph is established by Cockayne, Dawes, and Hedetniemi [2].