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Strong edge-coloring of planar graphs

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ABSTRACT

A strong edge-coloring of a graph is a proper edge-coloring where the edges at distance at most 2 receive distinct colors. It is known that every planar graph G has a strong edge-coloring with at most $4\,\Delta(G)+4$ colors. We show that $3\,\Delta(G)+5$ colors suffice if G has girth 6, and $3\,\Delta(G)$ colors suffice if its girth is at least 7. Moreover, we show that cubic planar graphs with girth at least 6 can be strongly edge-colored with at most nine colors.

1. Introduction

A *strong edge-coloring* of a graph G is a proper edge-coloring where every color class induces a matching, i.e., every two edges at distance at most 2 receive distinct colors. The smallest number of colors for which a strong edge-coloring of a graph G exists is called the *strong chromatic index*, $\chi_s'(G)$. In 1985, Erdős and Nešetřil posed the following conjecture during a seminar in Prague (later published in [2]).

Conjecture 1 (*Erdős, Nešetřil*)**.** *For every graph G,*

$$\chi_s'(G) \le \begin{cases} \frac{5}{4}\Delta(G)^2\,, & \Delta(G) \text{ is even;} \\ \frac{1}{4}(5\Delta(G)^2 - 2\Delta(G) + 1)\,, & \Delta(G) \text{ is odd.} \end{cases}$$

They also presented a construction showing that Conjecture 1, if true, is tight. In 1997, Molloy and Reed [9] established currently the best known upper bound for the strong chromatic index of graphs with sufficiently large maximum degree.

Theorem 2 (*Molloy, Reed*)**.** *For every graph G with sufficiently large maximum degree,*

$$\chi_s'(G) \le 1.998\,\Delta(G)^2.$$

In 1990, Faudree et al. [3] proposed several conjectures regarding subcubic graphs, i.e. graphs with maximum degree 3.

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Conjecture 3 (*Faudree et al.*)**.** *If G is a subcubic graph, then*

1. $\chi'_s(G) \leq 10$*;*
2. $\chi'_s(G) \leq 9$ *if G is bipartite;*
3. $\chi'_s(G) \leq 9$ *if G is planar;*
4. $\chi'_s(G) \leq 6$ *if G is bipartite and the weight of each edge uv is at most* 5*, i.e. at least one of its two endvertices has degree at most* 2*;*
5. $\chi'_s(G) \leq 7$ *if G is bipartite of girth* 6*;*
6. $\chi'_s(G) \leq 5$ *if G is bipartite and has girth large enough.*

Andersen [1] and independently Horák, Qing, and Trotter [7] confirmed that Conjecture 1 holds for subcubic graphs, i.e., that the strong chromatic index of any subcubic graph is at most 10, which solves also the first item of Conjecture 3. The second item of Conjecture 3 was confirmed by Steger and Yu [10].

In this paper, we consider planar graphs with lower bounds on girth. In 1990, Faudree et al. [3] found a construction of planar graphs showing that for every integer k with $k \geq 2$ there exists a planar graph G with maximum degree k and $\chi'_s(G) = 4k - 4$. Moreover, they proved the following theorem.

Theorem 4 (*Faudree et al.*)**.** *If G is a planar graph, then*

$$\chi'_s(G) \leq 4\,\chi'(G).$$

The proof of Theorem 4 is short and simple, so we include it.

Proof. Let M_i be the set of edges having color i. Let $G(M_i)$ be the graph obtained from G by contracting every edge of M_i. Note that the vertices corresponding to the edges of M_i that are incident to a common edge are adjacent in $G(M_i)$. Since $G(M_i)$ is planar, we can color the vertices with four colors by the Four Color Theorem, and therefore any two edges of M_i incident to a common edge receive distinct colors in G. After coloring $G(M_1), \ldots, G(M_k)$, where k is the chromatic index of G, we obtain a strong edge-coloring of G. □

By Vizing's Theorem [11], every graph G is $(\Delta(G)+1)$-edge-colorable and moreover, if $\Delta(G) \geq 7$, then $\Delta(G)$ colors suffice [12]. Hence, by Theorem 4, the strong chromatic index of every planar graph G is at most $4\Delta(G)+4$ or even at most $4\Delta(G)$, if $\Delta(G)$ is at least 7.

Contracting the edges of a matching reduces face-lengths by at most a factor of 2. We use this fact to apply Grötzsch's Theorem [4] to the idea of Theorem 4. Moreover, we combine the result of Kronk, Radlowski, and Franen [8], who showed that if a planar graph has $\Delta(G)$ at least 4 and girth at least 5, then its chromatic index equals $\Delta(G)$, and our result in Theorem 9 to obtain the following.

Theorem 5. *If G is a planar graph with girth at least* 7*, then $\chi'_s(G) \leq 3\,\Delta(G)$.*

Recently, Hocquard and Valicov [6] considered graphs with bounded maximum average degree. As a corollary they obtained the following results regarding planar graphs.

Theorem 6 (*Hocquard, Valicov*)**.** *If G is a planar subcubic graph with girth g, then*

(i) *if $g \geq 30$, then $\chi'_s(G) \leq 6$;*
(ii) *if $g \geq 11$, then $\chi'_s(G) \leq 7$;*
(iii) *if $g \geq 9$, then $\chi'_s(G) \leq 8$;*
(iv) *if $g \geq 8$, then $\chi'_s(G) \leq 9$.*

Later, these bounds were improved in [5] to the following:

Theorem 7 (*Hocquard et al.*)**.** *If G is a planar subcubic graph with girth g, then*

(i) *if $g \geq 14$, then $\chi'_s(G) \leq 6$;*
(ii) *if $g \geq 10$, then $\chi'_s(G) \leq 7$;*
(iii) *if $g \geq 8$, then $\chi'_s(G) \leq 8$;*
(iv) *if $g \geq 7$, then $\chi'_s(G) \leq 9$.*

In this paper we consider planar graphs with girth 6 and obtain the following results.

Theorem 8. *If G is a planar graph with girth at least* 6 *and maximum degree at least* 4*, then $\chi'_s(G) \leq 3\,\Delta(G)+5$.*

Theorem 9. *If G is a subcubic planar graph with girth at least* 6*, then $\chi'_s(G) \leq 9$.*

Theorem 9 partially solves the third item of Conjecture 3. The proposed bound, if true, is realized by the complement of C_6 (see Fig. 1). Here, let us remark that very recently Hocquard et al. [5] obtained an improved result of Theorem 9, proving that every subcubic planar graph without cycles of length 4 and 5 admits a strong edge-coloring with at most 9 colors.

All the graphs considered in the paper are simple. We say that a vertex of degree k, at least k, or at most k is a *k-vertex*, a *k^+-vertex*, or a *k^--vertex*, respectively. Similarly, a *k-neighbor*, a *k^+-neighbor*, or a *k^--neighbor* of a vertex v is a neighbor of v of degree k, at least k, or at most k, respectively. The 2-*neighborhood* of an edge e consists of the edges at distance at most 2 from e. Here, the distance between the edges e and e' in a graph G is defined as the distance between the vertices corresponding to e and e' in the line graph $L(G)$.