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On a class of nilpotent table algebras and applications to association schemes

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ABSTRACT

In this paper we present sufficient conditions under which a nilpotent table algebra is exactly isomorphic to the wreath product of thin table algebras, and show by examples that this result is not true when the conditions are replaced by weaker ones. Applications to association schemes are also discussed.

1. Introduction

The wreath product of table algebras provides an important way to construct the new table algebras from the old ones (cf. [2]). A special class of varietal Terwilliger algebras is characterized by their base C-algebras as wreath products of 2-dimensional and thin table algebras (cf. [10, Theorem 1.10]). Nilpotent table algebras are an important class of table algebras. In this paper we present sufficient conditions under which a nilpotent table algebra is exactly isomorphic to the wreath product of thin table algebras (see Theorem 1.1). As a corollary of [3, Theorem 3.2], Bagherian [3] proved the similar result for nilpotent association schemes (cf. [3, Corollary 3.6]). However, Example 3.5 shows that [3, Theorem 3.2] is not true. Our method in this paper is totally different from that in [3]. We will also present examples showing that Theorem 1.1 is not true when the conditions are replaced by weaker ones. As an immediate consequence of Theorem 1.1, we obtain the similar result for nilpotent association schemes (see Corollary 1.3), which is slightly stronger than [3, Corollary 3.6].

In the following we state the main results of the paper. Let us first state some necessary definitions and notation.

A *table algebra* $(A, \mathbf{B})$ is a finite dimensional associative algebra A over the complex numbers $\mathbb{C}$, and a distinguished basis $\mathbf{B} = \{b_0 = 1_A, b_1, b_2, \ldots, b_k\}$ for A such that the following properties hold:

(i) The structure constants for $\mathbf{B}$ are nonnegative real numbers; that is, for all $b_i, b_j \in \mathbf{B}$, $b_ib_j = \sum_{m=0}^{k} \lambda_{ijm} b_m$, for some $\lambda_{ijm} \in \mathbb{R}_{\geq 0}$.
(ii) There is an algebra antiautomorphism (denoted by $*$) of A such that $(a^*)^* = a$ for all $a \in A$ and $b_i^* \in \mathbf{B}$ for all $b_i \in \mathbf{B}$. (Hence i^* is defined by $b_{i^*} = b_i^*$.)
(iii) For all $b_i, b_j \in \mathbf{B}$, $\lambda_{ij0} = 0$ if $j \neq i^*$; and $\lambda_{ii^*0} > 0$.

Let $(A, \mathbf{B})$ be a table algebra. It is well known that $(A, \mathbf{B})$ has a (unique) *degree map* $\nu : A \to \mathbb{C}$ such that $\nu(b_i) = \nu(b_i^*) > 0$ for all $b_i \in \mathbf{B}$ (see [1]). The *order* of any $b_i \in \mathbf{B}$ is $o(b_i) := \nu(b_i)^2/\lambda_{ii^*0}$, and the *order* of any nonempty subset $\mathbf{N}$ of $\mathbf{B}$ is $o(\mathbf{N}) := \sum_{b_i \in \mathbf{N}} o(b_i)$. If for any $b_i \in \mathbf{B}$, $\nu(b_i) = \lambda_{ii^*0}$, then $(A, \mathbf{B})$ is called a *standard table algebra*. If all structure constants and degrees $\nu(b_i)$ are nonnegative integers, then $(A, \mathbf{B})$ is called an *integral table algebra*. Let $\mathbf{B} = \{b_0 = 1_A, b_1, b_2, \ldots, b_k\}$. For any $a \in A$

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with $a = \sum_{i=0}^{k} \alpha_i b_i$, define $\mathrm{Supp}(a) := \{b_i \mid \alpha_i \neq 0\}$. For any nonempty subsets $\mathbf{R}$ and $\mathbf{L}$ of $\mathbf{B}$, define $\mathbf{RL} := \bigcup_{b\in\mathbf{R}, c\in\mathbf{L}} \mathrm{Supp}(bc)$, and $\mathbf{R}^* = \{b^* \mid b \in \mathbf{R}\}$. A nonempty subset $\mathbf{N}$ of $\mathbf{B}$ is called a *closed subset* if $\mathbf{N}^*\mathbf{N} \subseteq \mathbf{N}$. If $\mathbf{N}$ is a closed subset of $\mathbf{B}$, then $1_A \in \mathbf{N}$, $\mathbf{N}^* = \mathbf{N}$, and $(\mathbb{C}\mathbf{N}, \mathbf{N})$ is also a table algebra, called a *table subalgebra* of $(A, \mathbf{B})$, where $\mathbb{C}\mathbf{N}$ is the $\mathbb{C}$-space with basis $\mathbf{N}$.

Let $(A, \mathbf{B})$ be a standard table algebra, and $\mathbf{N}$ a closed subset of $\mathbf{B}$. For any $b_i \in \mathbf{B}$, we write $\mathbf{N}\{b_i\}\mathbf{N}$ as $\mathbf{N}b_i\mathbf{N}$. Then $\{\mathbf{N}b_i\mathbf{N} \mid b_i \in \mathbf{B}\}$ forms a partition of $\mathbf{B}$. For any nonempty subset $\mathbf{R}$ of $\mathbf{B}$, let $\mathbf{R}^+ = \sum_{b_i\in\mathbf{R}} b_i$. Let

$$\mathbf{B}//\mathbf{N} := \{b_i//\mathbf{N} \mid b_i \in \mathbf{B}\}, \quad \text{where } b_i//\mathbf{N} := o(\mathbf{N})^{-1}(\mathbf{N}b_i\mathbf{N})^+. \tag{1.1}$$

Let $A//\mathbf{N} := \mathbb{C}(\mathbf{B}//\mathbf{N})$, the $\mathbb{C}$-space with basis $\mathbf{B}//\mathbf{N}$. Then $(A//\mathbf{N}, \mathbf{B}//\mathbf{N})$ is a standard table algebra, called the *quotient table algebra* of $(A, \mathbf{B})$ with respect to $\mathbf{N}$. Note that for any $b_i \in \mathbf{B}$, $(b_i//\mathbf{N})^* = b_i^*//\mathbf{N}$, and the order $o(b_i//\mathbf{N}) = o(\mathbf{N})^{-1}o(\mathbf{N}b_i\mathbf{N})$. (See [1, Theorem 4.9].) If A is commutative, we write $b_i//\mathbf{N}$ as $b_i/\mathbf{N}$, $\mathbf{B}//\mathbf{N}$ as $\mathbf{B}/\mathbf{N}$, and $A//\mathbf{N}$ as $A/\mathbf{N}$.

Let $(A, \mathbf{B})$ be a table algebra. An element $b_i \in \mathbf{B}$ is called a *thin* (or *linear*) *element* if $b_i b_i^* = \lambda_{ii^*0} 1_A$. The set of all thin elements of $\mathbf{B}$ is denoted by $L(\mathbf{B})$. It is known that $L(\mathbf{B})$ is a closed subset of $\mathbf{B}$. If $L(\mathbf{B}) = \mathbf{B}$, then $(A, \mathbf{B})$ is called a *thin* table algebra. If $(A, \mathbf{B})$ is standard, then define $L^{(i)}(\mathbf{B})$ for all $i \geq 0$ as follows: $L^{(0)}(\mathbf{B}) := \{1_A\}$, $L^{(1)}(\mathbf{B}) := L(\mathbf{B})$, and recursively, $L^{(i+1)}(\mathbf{B})$ is the closed subset of $\mathbf{B}$ such that $L^{(i+1)}(\mathbf{B})//L^{(i)}(\mathbf{B}) = L(\mathbf{B}//L^{(i)}(\mathbf{B}))$. If $(A, \mathbf{B})$ is standard and commutative, and $L^{(n)}(\mathbf{B}) = \mathbf{B}$ for some $n > 0$, then $(A, \mathbf{B})$ is called a *nilpotent table algebra*, and the minimal $n > 0$ such that $L^{(n)}(\mathbf{B}) = \mathbf{B}$ is called the *nilpotent class* of $\mathbf{B}$. For properties of nilpotent table algebras, the reader is referred to [5,11].

Let $(A, \mathbf{B})$ and $(C, \mathbf{D})$ be standard table algebras, with $\mathbf{B} = \{b_0 = 1_A, b_1, \ldots, b_k\}$ and $\mathbf{D} = \{d_0 = 1_C, d_1, \ldots, d_l\}$. Then the *tensor product* $(A \otimes C, \mathbf{B} \otimes \mathbf{D})$ is also a standard table algebra, where $\mathbf{B} \otimes \mathbf{D} := \{b_i \otimes d_j \mid 0 \leq i \leq k, 0 \leq j \leq l\}$. Let $\mathbf{B} \wr \mathbf{D} := \{b_0 \otimes d_j \mid 0 \leq j \leq l\} \cup \{b_i \otimes \mathbf{D}^+ \mid 1 \leq i \leq k\}$, and $A \wr C$ the $\mathbb{C}$-space with basis $\mathbf{B} \wr \mathbf{D}$. Then $(A \wr C, \mathbf{B} \wr \mathbf{D})$ is also a standard table algebra, called the *wreath product* of $(A, \mathbf{B})$ and $(C, \mathbf{D})$.

Two table algebras $(A, \mathbf{B})$ and $(C, \mathbf{D})$ are said to be *exactly isomorphic* and denoted by $(A, \mathbf{B}) \cong_x (C, \mathbf{D})$ or simply $\mathbf{B} \cong_x \mathbf{D}$, if there is an algebra isomorphism $\phi : A \to C$ such that $\phi(\mathbf{B}) = \mathbf{D}$, where $\phi(\mathbf{B}) := \{\phi(b_i) \mid b_i \in \mathbf{B}\}$.

The next theorem is the main result of the paper.

Theorem 1.1. *Let $(A, \mathbf{B})$ be a nilpotent table algebra of class n (≥ 2) such that*

$$\left|L^{(i+1)}(\mathbf{B})/L^{(i)}(\mathbf{B})\right| \text{ are distinct prime numbers,} \quad \text{where } 0 \leq i \leq n-2. \tag{1.2}$$

Then

$$\mathbf{B} \cong_x \left(\mathbf{B}/L^{(n-1)}(\mathbf{B})\right) \wr \left(L^{(n-1)}(\mathbf{B})/L^{(n-2)}(\mathbf{B})\right) \wr \cdots \wr \left(L^{(2)}(\mathbf{B})/L^{(1)}(\mathbf{B})\right) \wr L^{(1)}(\mathbf{B}) \tag{1.3}$$

if and only if $\left(A/L^{(i)}(\mathbf{B}), \mathbf{B}/L^{(i)}(\mathbf{B})\right)$ are all integral table algebras, $0 \leq i \leq n-2$.

Remark 1.2. (i) In the above theorem, we do not need any assumption about $\left|\mathbf{B}/L^{(n-1)}(\mathbf{B})\right|$.

(ii) Let $(A, \mathbf{B})$ be a nilpotent table algebra of class n (≥ 2) such that (1.3) holds, then $\left(A/L^{(i)}(\mathbf{B}), \mathbf{B}/L^{(i)}(\mathbf{B})\right)$ are all integral table algebras, $0 \leq i \leq n-2$, whether (1.2) is satisfied or not.

Let $\mathfrak{X} = (X, \{R_i\}_{0\leq i\leq d})$ be an association scheme. Let A_i be the adjacency matrix of R_i, $0 \leq i \leq d$, and let $(A, \mathbf{B})$ be the Bose–Mesner algebra of $\mathfrak{X}$, where $\mathbf{B} := \{A_0, A_1, \ldots, A_d\}$. Then $(A, \mathbf{B})$ is a standard table algebra. If $(A, \mathbf{B})$ is a nilpotent table algebra of class n, then $\mathfrak{X}$ is called a *nilpotent association scheme of class n*. As an immediate consequence of Theorem 1.1, we have the following corollary. Since there are different definitions of wreath products of association schemes in the literature, the definition of the wreath product of association schemes in the next corollary will be stated in the next section. For the definition of the isomorphism of association schemes, the reader is referred to [15].

Corollary 1.3. *Let $\mathfrak{X}$ be a nilpotent association scheme of class n such that its Bose–Mesner algebra $(A, \mathbf{B})$ satisfies (1.2). Then $\mathfrak{X}$ is isomorphic to the wreath product of thin association schemes.*

Since there is no assumption about $|\mathbf{B}/L^{(n-1)}(\mathbf{B})|$, the above corollary is slightly stronger than [3, Corollary 3.6].

The wreath products of association schemes have been studied in several papers; for example, see [8,9]. In particular, assume that $\mathfrak{X}$ is a nilpotent association scheme. Then by [9, Theorem B], $\mathfrak{X}$ is isomorphic to the wreath product of thin association schemes if and only if for any adjacency matrices A_i, A_j such that $A_i \neq A_j^*$, where A_j^* is the transpose of A_j, $|\mathrm{supp}(A_iA_j)| = 1$.

Theorem 1.1 and Corollary 1.3 are proved in Section 2. In Section 3 we present a counterexample to [3, Theorem 3.2] as well as examples showing that (1.3) is not true if (1.2) is replaced by weaker conditions.

2. Proofs of Theorem 1.1 and Corollary 1.3

Let us first prove Theorem 1.1. We need the next two lemmas.

Lemma 2.1 (*cf.* [12, *Lemmas 3.1 and 3.3*]). *Let $(A, \mathbf{B})$ be a standard table algebra, and $\mathbf{N}$ a closed subset of $\mathbf{B}$. Then the following hold.*

(i) *$\mathbf{B} \cong_x (\mathbf{B}//\mathbf{N}) \wr \mathbf{N}$ if and only if for any $b_i \in \mathbf{N}$ and $b_j \in \mathbf{B} \setminus \mathbf{N}$, $b_ib_j = b_jb_i = o(b_i)b_j$.*
(ii) *If $(A, \mathbf{B}) \cong_x (A//\mathbf{N}, \mathbf{B}//\mathbf{N}) \wr (U, \mathbf{V})$ for some standard table algebra $(U, \mathbf{V})$, then $(\mathbb{C}\mathbf{N}, \mathbf{N}) \cong_x (U, \mathbf{V})$ and $\mathbf{B} \cong_x (\mathbf{B}//\mathbf{N}) \wr \mathbf{N}$.*
(iii) *If $(A, \mathbf{B}) \cong_x (C, \mathbf{D}) \wr (\mathbb{C}\mathbf{N}, \mathbf{N})$ for some standard table algebra $(C, \mathbf{D})$, then $(C, \mathbf{D}) \cong_x (A//\mathbf{N}, \mathbf{B}//\mathbf{N})$ and $\mathbf{B} \cong_x (\mathbf{B}//\mathbf{N}) \wr \mathbf{N}$.*