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Long properly colored cycles in edge colored complete graphs

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ABSTRACT

Let K_n^c denote a complete graph on n vertices whose edges are colored in an arbitrary way. Let $\Delta^{\text{mon}}(K_n^c)$ denote the maximum number of edges of the same color incident with a vertex of K_n^c. A properly colored cycle (path) in K_n^c is a cycle (path) in which adjacent edges have distinct colors. B. Bollobás and P. Erdös (1976) proposed the following conjecture: if $\Delta^{\text{mon}}(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then K_n^c contains a properly colored Hamiltonian cycle. Li, Wang and Zhou proved that if $\Delta^{\text{mon}}(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then K_n^c contains a properly colored cycle of length at least $\lceil \frac{n+2}{3} \rceil + 1$. In this paper, we improve the bound to $\lceil \frac{n}{2} \rceil + 2$.

1. Introduction

All graphs considered here are finite, undirected, and simple. Let G be a graph with vertex set V and edge set E.

An *edge coloring* of a graph is an assignment of "colors" to the edges of the graph. An *edge colored graph* is a graph with an edge coloring. A cycle (path) in an edge colored graph is *properly colored* if no two adjacent edges in it have the same color.

Grossman and Häggkvist [9] gave a sufficient condition on the existence of a properly colored cycle in edge colored graphs with two colors, and Yeo [17] extended the result to edge colored graphs with any number of colors. Given an edge colored graph G, let $\deg^c(v)$, named the color degree of a vertex v, be defined as the maximum number of edges incident to v that have distinct colors. The minimum color degree $\delta^c(G)$ is the minimum $\deg^c(v)$ over all vertices of G. In [12], some minimum color degree conditions for the existence of properly colored cycles are obtained. In particular, they proved that if G is an edge colored graph of order n satisfying $\delta^c(G) \geq d \geq 2$, then either G has a properly colored path of length at least $2d$, or G has a properly colored cycle of length at least $\lceil \frac{2d}{3} \rceil + 1$. In [13], Lo improved the bound $\lceil \frac{2d}{3} \rceil + 1$ to the best possible value $d + 1$.

Let K_n^c be an edge colored complete graph on n vertices, and let $c(u, v)$ be the color assigned to edge uv. In [4], Barr gave a simple sufficient condition for the existence of a properly colored Hamiltonian path in K_n^c: K_n^c has no monochromatic triangles. Bang-Jensen, Gutin and Yeo [3] proved that if K_n^c contains a properly colored 2-factor, then it has a properly colored Hamiltonian path. In [7], Feng et al. proved that K_n^c has a properly colored Hamilton path iff K_n^c has a properly colored almost 2-factor (an almost 2-factor is a spanning subgraph consisting of disjoint cycles and a path).

Bollobás and Erdös [5] proved that if $n \geq 3$ and $\delta^c(K_n^c) \geq \frac{7n}{8}$, then there exists a properly colored Hamiltonian cycle. They also proposed a question: whether the bound $\frac{7n}{8}$ can be improved to $\frac{n+5}{3}$. Fujita and Magnant [8] constructed an edge colored complete graph K_{2m} with $\delta^c(K_{2m}) = m$, which has no properly colored Hamiltonian cycles. So $\delta^c(K_n) = \lfloor \frac{n}{2} \rfloor$ is not

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enough. Let $\Delta^{\text{mon}}(K_n^c)$ be the maximum number of edges of the same color incident with a vertex of K_n^c. In the same paper, Bollobás and Erdös proposed the following conjecture.

Conjecture 1 (*Bollobás and Erdös [5]*). *If $\Delta^{\text{mon}}(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then K_n^c has a properly colored Hamiltonian cycle.*

Bollobás and Erdös proved that if $\Delta^{\text{mon}}(K_n^c) \leq \frac{n}{69}$ then K_n^c contains a properly colored Hamiltonian cycle. This result was improved by Chen and Daykin [6] and Shearer [15].

As far as we know, the best asymptotic estimate was obtained by Alon and Gutin using the probabilistic method.

Theorem 1.1 (*Alon and Gutin [1]*). *For every positive real number ϵ, there exists $n_0 = n_0(\epsilon)$ so that for every $n > n_0$, if K_n^c satisfies*

$$\Delta^{\text{mon}}(K_n^c) \leq \left(1 - \frac{1}{\sqrt{2}} - \epsilon\right) n, \tag{1}$$

then K_n^c contains a properly colored Hamiltonian cycle.

Li, Wang and Zhou [12] studied long properly colored cycles in edge colored complete graphs and proved that if $\Delta^{\text{mon}}(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then K_n^c contains a properly colored cycle of length at least $\lceil \frac{n+2}{3} \rceil + 1$. For more details concerning properly colored cycles and paths, we refer the reader to [2,11,10,16]. In this paper, we improve the bound on the length of the properly colored cycles and prove the following theorem.

Theorem 1.2. *If $\Delta^{\text{mon}}(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then K_n^c contains a properly colored cycle of length at least $\lceil \frac{n}{2} \rceil + 2$.*

The main idea is the rotation–extension technique of Pósa [14], which was used on general edge colored graphs in [13].

When a and b are integers, the notation $[a, b]$ is used to indicate the interval of all integers between a and b, including both. In particular, $[a, a-1]$ is an empty set. A properly colored path P of length ℓ is viewed as an ℓ-tuple $(v_0, v_1, \ldots, v_\ell)$, which is different from $(v_\ell, v_{\ell-1}, \ldots, v_0)$. For such a properly colored path $P = (v_1, v_2, \ldots, v_\ell)$, denote $\{v \mid c(v_1, v_2) \neq c(v_1, v)\}$ by $N^c(v_1, P)$ and $\{v \mid c(v_\ell, v_{\ell-1}) \neq c(v, v_\ell)\}$ by $N^c(v_\ell, P)$.

2. The proof of Theorem 1.2

If $3 \leq n \leq 5$, then K_n^c is a properly colored complete graph, and hence each Hamiltonian cycle is properly colored. Our conclusion holds clearly. So we may assume that $n \geq 6$.

We will prove Theorem 1.2 by contradiction. Suppose that each properly colored cycle is of length at most $\lceil \frac{n}{2} \rceil + 1$. For simplicity, let the vertices of K_n^c be labeled with integers from 1 to n. Let P_0 be a longest properly colored path. Without loss of generality, assume that $P_0 = (1, 2, \ldots, \ell)$.

Put

$$X = \{x_1, x_2, \ldots, x_p\} = \{x_i \mid c(1, x_i) \neq c(1, 2)\} = N^c(1, P_0)$$

and

$$Y = \{y_1, y_2, \ldots, y_q\} = \{y_i \mid c(y_i, \ell) \neq c(\ell - 1, \ell)\} = N^c(\ell, P_0).$$

Moreover, $x_1, x_2, \ldots, x_p$ and $y_1, y_2, \ldots, y_q$ are increasing sequences. Since $\Delta^{\text{mon}}(K_n^c) < \lfloor \frac{n}{2} \rfloor$, we have $\min\{|X|, |Y|\} \geq \lceil \frac{n}{2} \rceil$; consequently, $x_1 < x_p$. Note that $X, Y \subset V(P_0)$ since P_0 is a longest properly colored path. Thus $\ell \geq \lceil \frac{n}{2} \rceil + 2$. Note that either $\ell \notin X$ or $1 \notin Y$; otherwise $(1, 2, \ldots, \ell, 1)$ is a properly colored cycle of length at least $\lceil \frac{n}{2} \rceil + 2$, which is a contradiction. Hence $\ell \geq \lceil \frac{n}{2} \rceil + 3$.

Let $y_s \in Y$ be the maximum such that $c(\ell, y_i) = c(y_i, y_i + 1)$ for all $y_i \in [y_1, y_s] \cap Y$. Note that y_s is well defined, since $c(\ell, y_1) = c(y_1, y_1 + 1)$; otherwise, $(y_1, y_1 + 1, \ldots, \ell, y_1)$ is a properly colored cycle of length at least $\lceil \frac{n}{2} \rceil + 2$. Clearly, $y_s \leq \ell - 2$.

Note that $c(1, x_p) = c(x_p - 1, x_p)$ or else $(1, 2, \ldots, x_p, 1)$ is a properly colored cycle of length at least $\lceil \frac{n}{2} \rceil + 2$, a contradiction.

Claim 1. $y_s \leq x_p - 3$.

Proof of Claim 1. If $x_p = \ell$, then $y_s \leq \ell - 3$; otherwise, $(1, 2, \ldots, \ell - 2, \ell, 1)$ is a properly colored cycle of length $\ell - 1 \geq \lceil \frac{n}{2} \rceil + 2$, a contradiction. So we may assume that $x_p \leq \ell - 1$. Let $y_i \in Y$ be the maximum such that $y_i < x_p$. If $c(y_i, y_i+1) = c(y_i, \ell)$, then $(1, \ldots, y_i, \ell, \ell-1, \ldots, x_p, 1)$ is a properly colored cycle containing $Y \cup \{\ell, \ell-1\}$, a contradiction. Hence, $c(y_i, y_i+1) \neq c(y_i, \ell)$, by the definition of y_s, we have $y_s \leq x_p-2$. If $y_s = x_p-2$, then $(1, \ldots, x_p-2, \ell, \ell-1, \ldots, x_p, 1)$ is a properly colored cycle of length $\ell - 1 \geq \lceil \frac{n}{2} \rceil + 2$, which is a contradiction. Therefore, $y_s \leq x_p - 3$. □

Claim 2. *There exist $u, w \in X$ such that $1 \leq y_1 \leq y_s < u \leq \lceil \frac{n}{2} \rceil + 1$, $u < w$ and the following proposition holds:*

(a) *$c(y_i, \ell) = c(y_i, y_i + 1)$ for all $y_i \in [y_1, y_s] \cap Y$;*