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Binary matroids with no 4-spike minors

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ABSTRACT

For a simple binary matroid M having no n-spike minor, we examine the problem of bounding $|E(M)|$ as a function of its rank $r(M)$ and circumference $c(M)$. In particular, we show that $|E(M)| \leq \min\left\{\frac{r(M)(r(M)+3)}{2},\ c(M)r(M)\right\}$ for any simple, binary matroid M having no 4-spike minor. As a consequence, the same bound applies to simple, binary matroids having no $AG(3, 2)$-minor.

1. Introduction

This paper concerns a problem in *extremal matroid theory*, an area of matroid theory which, much like extremal graph theory, is rich with numerous problems. For two surveys on this subject, see [2,8]. One general problem is, given a certain minor-closed class of matroids, find good bounds, as a function of rank, for the maximum size of a simple matroid in this class. Of particular interest are classes of binary matroids with certain forbidden minors. For $k \geq 2$, let $\mathscr{P}_k$ be the class of binary matroids having no $PG(k, 2)$-minor. Finding sharp bounds for $\mathscr{P}_k$ is an extremely hard problem, although we have the following good bound for the class $\mathscr{P}_2$ due to Heller [7] and Murty [12]:

Theorem 1.1 (*Heller, Murty*)**.** *If $M \in \mathscr{P}_2$ is simple, then $|E(M)| \leq \binom{r(M)+1}{2}$.*

For binary minor-closed classes of matroids which contain $\mathscr{P}_2$, the problem of bounding matroid size becomes quite tricky. In [9], Kung showed that $|E(M)| \leq \frac{15}{4}\binom{r(M)+1}{2}$ for all simple $M \in \mathscr{P}_3$. When considering larger classes containing $\mathscr{P}_2$, it is useful to consider single element extensions of F_7, for example, the affine geometry $AG(3, 2)$. One may ask, what bounds are possible for the class of binary matroids containing no $AG(3, 2)$-minor? Another natural minor-closed class of matroids to consider is the class of binary matroids with no n-spike minor. An *n-spike* is a simple matroid M consisting of n triangles $T_1, \ldots, T_n$ meeting at a point e where

(i) $r(T_1 \cup \cdots \cup T_k) = k + 1,\ k = 1, \ldots, n - 1$

and

(ii) $r(T_1 \cup \cdots \cup T_n) = n$.

Note that a binary 3-spike is the Fano plane F_7. N-spikes appear in numerous places in the literature on matroids: for example, see [3,5,14,15]. In particular, they are important in the theory of matroid representation. It was shown in [14] that n-spikes can have an unbounded number of inequivalent $GF(q)$-representations for $q \geq 7$. For $k \geq 3$, let $\mathscr{N}_k$ be the set

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of binary matroids having no k-spike minor. Then $\mathscr{P}_2 \subset \mathscr{N}_k$ for all $k \geq 3$. It is also easy to show that every non-trivial binary extension of the affine geometry $AG(3, 2)$ is either isomorphic to a 4-spike or its dual (see Oxley [13]). For this reason, the class $\mathscr{N}_4$ is interesting and relevant here. In this paper, we focus on finding good bounds for the maximum size of simple matroids in $\mathscr{N}_4$. It should be mentioned that "growth rates" of classes like $\mathscr{N}_k$ and $\mathscr{P}_k$ are bounded by quadratic functions of rank. More generally, let $\mathscr{M}$ be a minor-closed class of matroids. The following theorem due to Geelen, Kung, and Whittle [6] demonstrates that the growth of the size of the matroids in $\mathscr{M}$, when it is finite, is either linear, quadratic, or exponential in terms of the rank.

Theorem 1.2 (*Geelen, Kung, Whittle*)**.** *Let $\mathscr{M}$ be a minor-closed class of matroids. Then one of the four possibilities below holds:*

(i) *There exists $\alpha > 0$ such that $|E(M)| \leq \alpha r(M)$ for all simple $M \in \mathscr{M}$.*
(ii) *$\mathscr{M}$ contains all graphic matroids and there exists $\alpha > 0$ such that $|E(M)| \leq \alpha r(M)^2$ for all simple $M \in \mathscr{M}$.*
(iii) *For some power of q, $\mathscr{M}$ contains all $GF(q)$-representable matroids, and $|E(M)| \leq \alpha q^{r(M)}$ for some $\alpha > 0$ and for all $M \in \mathscr{M}$.*
(iv) *$\mathscr{M}$ contains all rank two matroids.*

As a consequence of the above theorem, there exist constants $\alpha(k) > 0$ and $\beta(k) > 0$ such that $|E(M)| \leq \alpha(k)r(M)^2$ for all simple $M \in \mathscr{N}_k$, and $|E(M)| \leq \beta(k)r(M)^2$ for all simple $M \in \mathscr{P}_k$. Let $c(M)$ denote the *circumference* of a matroid M, the size of a largest circuit. The main result of this paper, which we shall prove in the next two sections, is the following bound for $\mathscr{N}_4$:

Theorem 1.3. *Let M be a simple, binary matroid having no 4-spike minor. Then*

$$|E(M)| \leq \min\left\{\frac{r(M)(r(M)+3)}{2},\ c(M)r(M)\right\}.$$

Since every 4-spike contains $AG(3, 2)$ as a minor, $\mathscr{N}_4$ contains the class of matroids having no $AG(3, 2)$-minor, and consequently, we have the following corollary to the above theorem:

Corollary 1.4. *Let M be a simple, binary matroid having no $AG(3, 2)$-minor. Then*

$$|E(M)| \leq \min\left\{\frac{r(M)(r(M)+3)}{2},\ c(M)r(M)\right\}.$$

In a paper which has only recently come to the attention of the author, Kung et al. [10] showed that for any simple, binary matroid M having no $AG(3, 2)$-minor, $|E(M)| \leq \binom{r(M)+1}{2}$, when $r(M) \geq 5$. In addition, they also show that for $r(M) = r$, the matroid $M(K_{r+1})$ is the unique matroid meeting the bound when $r \geq 6$. Their proof uses a computer check to verify the above bound for matroids of rank 5.

The bound in Theorem 1.1 can be improved using the following bound involving the circumference:

Theorem 1.5 (*McGuinness [11]*)**.** *Let M be a simple, binary matroid having no F_7-minor. Then $|E(M)| \leq \frac{1}{2}c(M)r(M)$.*

It is worth remarking that the above bound extends a well-known result of Erdős and Gallai (see [1,4]), which states that for a simple graph G on n vertices having circumference c, $|E(G)| \leq \frac{1}{2}c(n-1)$. We believe that similar bounds are possible for the class $\mathscr{P}_k$ in general: we venture the following conjecture:

Conjecture 1.6. *For all $k \geq 2$ there is a constant $\alpha(k) > 0$ depending only on k such that $|E(M)| \leq \alpha(k)c(M)r(M)$ for all simple matroids $M \in \mathscr{P}_k$.*

For $\mathscr{P}_3$, it seems possible to achieve the bound $|E(M)| \leq \frac{15}{4}r(M)c(M)$ by changing Kung's proof in [9], although I have not worked through all the details. In Section 5, we show that the above conjecture holds with $\mathscr{N}_k$ in place of $\mathscr{P}_k$. In Section 4, we show that growth rate bounds for the class $\mathscr{N}_k$ is related to the growth rate for the class of binary matroids having circumference at most $k-1$.

2. Bounding the number of triangles

We shall refer to a circuit with two elements as a *digon*. A *Hamilton circuit* in a matroid is a spanning circuit. An element e is called a *chord* of a circuit C if $e \in \mathrm{cl}(C) \setminus C$. In this section, we shall prove that if $M \in \mathscr{N}_4$ is simple and has a Hamilton circuit C, then there are at most $|C| - 1$ triangles which contain a fixed element $e \in C$. It turns out that this is the key ingredient for the proof of Theorem 1.3. To do this, we shall need the following useful observation:

Lemma 2.1. *Let M be a simple binary matroid and let $T_1, \ldots, T_n$ be triangles containing an element e. Let $e_1, \ldots, e_n$ be distinct elements where $e_i \in T_i \setminus e$, $i = 1, \ldots, n$ and let $M_1 = si(M/e)$. Assuming $e_1, \ldots, e_n \in E(M_1)$, if there is a circuit in M_1 containing k of the elements $e_1, \ldots, e_n$, then M has a k-spike minor.*