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Backbone coloring for graphs with large girths

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ABSTRACT

For a graph G and a subgraph H (called a *backbone* graph) of G, a *backbone k-coloring of G with respect to H* is a proper vertex coloring of G using colors from the set $\{1, 2, \dots, k\}$, with an additional condition that colors for any two adjacent vertices in H must differ by at least two. The *backbone chromatic number of G over H*, denoted by $\mathrm{BBC}(G, H)$, is the smallest k of a backbone k-coloring admitted by G with respect to H. Broersma, Fomin, Golovach, and Woeginger (2007) [2] showed that $\mathrm{BBC}(G, H) \le 2\chi(G) - 1$ holds for every G and H; moreover, for every n there exists a graph G with a spanning tree T such that $\chi(G) = n$ and the bound is sharp. To answer a question raised in Broersma et al. (2007) [2], Miškuf, Škrekovski, and Tancer (2009) [16] proved that for any n there exists a triangle-free graph G with a spanning tree T such that $\chi(G) = n$ and $\mathrm{BBC}(G, T) = 2n - 1$. We extend this result by showing that for any positive integers n and l, there exists a graph G with a spanning tree T such that G has girth at least l, $\chi(G) = n$, and $\mathrm{BBC}(G, T) = 2n - 1$.

1. Introduction

Backbone coloring is a model for the channel assignment problem introduced by Hale [11]. The task in the channel assignment problem is to assign channels to a set of transmitters such that interference is avoided. Usually interference is divided into two types: strong interference and weak interference. The channels assigned to two transmitters with strong interference should be far apart, and channels assigned to two transmitters with weak interference should be distinct. A well studied graph theory model for the channel assignment problem is the distance-two labeling of graphs. We construct a graph where vertices represent the transmitters. If two vertices in the model graph are adjacent then stronger interference might occur between the two corresponding transmitters so the separation of these two channels needs to be at least two; and for two vertices that are distance two apart (that is, they are not adjacent but they share a common neighbor in G) then weak interference might occur between the two corresponding transmitters so they must receive different channels.

Backbone coloring of a graph is another model for the channel assignment problem, where edges in G are of two different types. Let H be a subgraph of G. An edge of G is either an edge of H which represents strong interference, or not an edge of H which represents weak interference. The subgraph H is called the *backbone* of G. In a backbone coloring of G with backbone H, colors assigned to a pair of vertices adjacent in H must be at least two apart, while vertices adjacent in G but not in H must get different colors. To be precise, a *backbone k-coloring of G with respect to H* is a function $f : V(G) \to \{1, 2, \dots, k\}$ such that the following are satisfied:

$$|f(u) - f(v)| \ge \begin{cases} 2 & \text{if } uv \in E(H); \\ 1 & \text{if } uv \in E(G) \setminus E(H). \end{cases}$$

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The *backbone chromatic number of G over H*, denoted by $\mathrm{BBC}(G, H)$, is the minimum k for which there is a backbone k-coloring of G with respect to H.

For a graph G, the *square of* G, denoted by G^2, has $V(G)$ as the vertex set and $uv \in E(G^2)$ if $uv \in E(G)$ or there is a 2-path from u to v in G. A *distance-two labeling* (also known as $L(2, 1)$-*labeling*) is the same as a backbone coloring of G^2 with respect to the backbone G (however, a distance-two labeling allows 0 as a color while a backbone coloring uses only positive integers, hence a distance-two k-labeling of a graph G is a backbone $(k+1)$-coloring of G^2 with respect to G). Introduced by Griggs and Yeh [10], distance-two labeling has been studied extensively in the past three decades (cf. [5–10,20,12–15,18,19,21]).

Backbone coloring was first introduced by Broersma et al. [1] and has been investigated widely by several authors in recent years (cf. [1–4,16,17]). Broersma, Fomin, Golovach, and Woeginger [2] studied the $\mathrm{BBC}(G, H)$-number when the backbone graph H is a spanning tree or a spanning path (if it exists) of G. Miškuf, Škrekovski, and Tancer [17] proved that for a graph G with maximum degree Δ and backbone H being a d-degenerated subgraph of G, then $\mathrm{BBC}(G, H) \le \Delta + d + 1$; moreover, if H is a matching then $\mathrm{BBC}(G, H) \le \Delta + 1$.

Denote the chromatic number of a graph G by $\chi(G)$. By properly coloring the vertices of G from the set $\{1, 3, 5, \ldots, 2\chi(G) - 1\}$, one obtains a backbone $(2\chi(G) - 1)$-coloring of G with respect to any subgraph H. Therefore, for any graph G and any subgraph H of G, $\mathrm{BBC}(G, H) \le 2\chi(G) - 1$. It was proved in [2] that for any positive integer n, there exists a graph G and a spanning tree T of G such that $\chi(G) = n$ and $\mathrm{BBC}(G, T) = 2\chi(G) - 1$. The graphs G used in the proof of this result are complete n-partite graphs, which contain many triangles.

An interesting question asked in [2] was whether there exists a constant c such that $\mathrm{BBC}(G, T) \le \chi(G) + c$ holds for all triangle-free graphs G and spanning tree T of G. Miškuf, Škrekovski, and Tancer [16] answered this question in the negative by showing that for any n there exists a triangle-free graph G with a spanning tree T such that $\chi(G) = n$ and $\mathrm{BBC}(G, T) = 2n-1$. The graphs constructed in [16], by a process similar to the construction of Mycielski graphs, are infinite and contain 4-cycles. Naturally, the authors raised the question regarding the existence of a graph G with large girth (i.e., the length of a shortest cycle in G) such that $\mathrm{BBC}(G, T) = 2\chi(G) - 1$ for some spanning tree T. We answer this question in positive.

Theorem 1. *For any positive integers n and l, there exists a graph G with girth greater than l and $\chi(G) = n$, and a spanning forest T of G such that* $\mathrm{BBC}(G, T) = 2n - 1$.

The proof of Theorem 1 is presented in the next two sections.

2. Construction of *G* and *T*

The construction of G and T in Theorem 1 will be based on the following result which seems to be a folklore. But we could not find a reference with the exact statement. For the completeness of the paper, we provide here an easy probabilistic proof.

Lemma 2. *For any positive integers n, l, m_0, for each $\delta > 0$, there is an n-partite graph G with partite sets $V_1, V_2, \ldots, V_n$ such that the following hold:*

- $|V_i| = m \ge m_0$.
- *G has $\chi(G) = n$ and girth greater than l.*
- *For any $1 \le i \ne j \le n$, for any $A \subset V_i$, $B \subset V_j$ with $|A|, |B| \ge \delta m$, there is an edge between A and B.*

Proof. Let $\epsilon = 1/(2l)$, m be a sufficiently large integer, and $p = m^{-1+\epsilon}$. Let G be a random graph with vertex set $V = V_1 \cup V_2 \cdots \cup V_n$, where $|V_i| = 2m$, and uv is an edge of G with probability p for any $u \in V_i$, $v \in V_j$ $(1 \le i < j \le n)$. Let X be the random variable which is the number of cycles in G of length at most l. The expectation of X is

$$E(X) \le \sum_{i=1}^{l} \frac{(2nm)_i}{2i} p^i \le l(2nmp)^l = l(2n)^l m^{1/2},$$

where $(x)_i = x(x-1)\cdots(x-i+1)$. Hence

$$P(X > m) \le \frac{E(X)}{m} \to 0, \quad \text{as } m \to \infty.$$

Let Y be the number of pairs of sets A, B such that $A \subset V_i$, $B \subset V_j$ with $i \ne j$, $|A|, |B| \ge \delta m$, and $E[A, B] = \emptyset$ (that is, there are no edges between A and B). The expectation of Y is

$$E(Y) \le 2^{2nm}(1-p)^{(\delta m)^2} \le e^{2nm - p\delta^2 m^2} = e^{2nm - \delta^2 m^{1+\epsilon}} \to 0, \quad \text{as } m \to \infty.$$

Hence

$$P(Y \ge 1) \le E(Y) \to 0, \quad \text{as } m \to \infty.$$

So for m sufficiently large, $P(X > m) < 1/2$ and $P(Y \ge 1) < 1/2$, and hence $P((X \le m) \wedge (Y = 0)) > 0$. This implies that there is a graph G in which the number of short cycles is less than m and for any pair $A \subset V_i$, $B \subset V_j$ with $i \ne j$, $|A|, |B| \ge \delta m$, we have $E[A, B] \ne \emptyset$. Delete m vertices from each V_i (for $i = 1, 2, \ldots, n$) so that each short cycle intersects the deleted vertices. The resulting graph G' has girth at least $l + 1$.