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Determinants of grids, tori, cylinders and Möbius ladders

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ABSTRACT

Recently, Bieñ [A. Bieñ, The problem of singularity for planar grids, Discrete Math. 311 (2011) 921–931] obtained a recursive formula for the determinant of a grid. Also, recently, Pragel [D. Pragel, Determinants of box products of paths, Discrete Math. 312 (2012) 1844–1847], independently, obtained an explicit formula for this determinant. In this paper, we give a short proof for this problem. Furthermore, applying the same technique, we get explicit formulas for the determinant of a torus, that of a cylinder, and that of a Möbius ladder.

1. Introduction and results

We denote by $A(G)$ the adjacency matrix of a graph G. A path (cycle) on n vertices is denoted by P_n (resp., C_n). Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their *Cartesian product* $G_1 \square G_2$ is the graph with vertex set $V_1 \times V_2$ and edge set

$$\Big\{\big((u, v), (u', v)\big) : (u, u') \in E_1, v \in V_2\Big\} \bigcup \Big\{\big((u, v), (u, v')\big) : u \in V_1, (v, v') \in E_2\Big\}.$$

The Cartesian product produces many important classes of graphs. For example, a *grid* (also called a *mesh*) is the Cartesian product of two paths, a *torus* (also called a *toroidal grid* or a *toroidal mesh*) is the Cartesian product of two cycles, and a *cylinder* is the Cartesian product of a path and a cycle. One can generalize these definitions to more than two paths or cycles. These classes of graphs are widely used computer architectures (e.g., grids are widely used in multiprocessor VLSI systems) [7].

The *nullity* of a graph G of order n, denoted by $\eta(G)$, is the multiplicity of 0 in the spectrum of G. Clearly, $\eta(G) = n - r(A(G))$, where $r(A(G))$ is the rank of $A(G)$. The nullity of a graph is closely related to the minimum rank problem for a family of matrices associated with a graph (see, e.g., [5] and the references therein). Nullity of a (molecular) graph (specifically, determining whether it is positive or zero) also has important applications in quantum chemistry and Hückel molecular orbital (HMO) theory (see, e.g., [6] and the references therein). A famous problem, posed by Collatz and Sinogowitz in 1957 [4], asks for the characterization of all graphs with positive nullity. Clearly, $\det A(G) = 0$ if and only if $\eta(G) > 0$. So, examining the determinant of a graph is a way to attack this problem. But there seems to be little published on calculating the determinants of various classes of graphs.

Recently, Bieñ [2] obtained a recursive formula for the determinant of a grid. Also, recently, Pragel [8], independently, obtained an explicit formula for this determinant (see (1) below). Here, using trigonometric identities, we give a short proof for this problem. Furthermore, applying the same technique, we get explicit formulas for the determinant of a torus, and that of a cylinder.

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Theorem 1. *Let $m > 1$ and $n > 1$ be integers. Then*

$$\det A(P_{m-1} \square P_{n-1}) = \begin{cases} (-1)^{\frac{(m-1)(n-1)}{2}} & \text{if } \gcd(m,n) = 1; \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

$$\det A(C_m \square C_n) = \begin{cases} 4^{\gcd(m,n)} & \text{if } m \text{ and } n \text{ are odd}; \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

$$\det A(P_{m-1} \square C_n) = \begin{cases} m & \text{if } n \text{ is odd and } \gcd(m,n) = 1; \\ (-1)^{m-1} m^2 & \text{if } n \text{ is even and } \gcd(m,n/2) = 1; \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

Note that for $m = 2$, (1) and (3) give the following well-known determinants:

$$\det A(P_{n-1}) = \begin{cases} (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd}; \\ 0 & \text{otherwise}. \end{cases} \quad \text{and} \quad \det A(C_n) = \begin{cases} 2 & \text{if } n \text{ is odd}; \\ -4 & \text{if } n \equiv 2 \pmod 4; \\ 0 & \text{otherwise}. \end{cases}$$

Our proof technique or its modifications may be useful in other situations with similar flavor (see, e.g., [1]). For example, let us consider the *Möbius ladder* M_{2n}, the graph on $2n$ vertices whose edge set is the union of the edge set of C_{2n} and $\{(v_i, v_{n+i}) : i = 1, \ldots, n\}$. We prove:

Theorem 2. *Let $n > 1$ be an integer. Then*

$$\det A(M_{2n}) = \begin{cases} -3 & \text{if } n \equiv \pm 2 \pmod 6; \\ -9 & \text{if } n \equiv \pm 1 \pmod 6; \\ 0 & \text{otherwise}. \end{cases} \tag{4}$$

2. Techniques and proofs

The starting point of our calculations is the following well-known theorem which gives the eigenvalues of the Cartesian product of two graphs (see, e.g., [9, p. 587]).

Theorem 3. *Let G_1 be a graph of order m, and G_2 be a graph of order n. If the eigenvalues of $A(G_1)$ and $A(G_2)$ are, respectively, $\lambda_1(G_1), \ldots, \lambda_m(G_1)$ and $\lambda_1(G_2), \ldots, \lambda_n(G_2)$, then the eigenvalues of $A(G_1 \square G_2)$ are precisely the numbers $\lambda_i(G_1) + \lambda_j(G_2)$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.*

We also need the following trigonometric identities, which might be of independent interest.

Lemma 4. *Let n be a positive integer and let $a \in \mathbb{Z}$ be such that $\gcd(a, n) = 1$. Then for any real number x,*

$$\sin(nx) = 2^{n-1}(-1)^{\frac{(a-1)(n-1)}{2}} \prod_{j=0}^{n-1} \sin\left(x + \frac{aj\pi}{n}\right). \tag{5}$$

Moreover,

$$\prod_{j=1}^{n-1} \sin\left(\frac{aj\pi}{n}\right) = (-1)^{\frac{(a-1)(n-1)}{2}} \cdot \frac{n}{2^{n-1}} \tag{6}$$

and

$$\prod_{j=1}^{n-1} \cos\left(\frac{aj\pi}{n}\right) = \begin{cases} (-1)^{\frac{a(n-1)}{2}} \cdot \frac{1}{2^{n-1}} & \text{if } n \text{ is odd}; \\ 0 & \text{otherwise}. \end{cases} \tag{7}$$

Proof. Let $\omega = e^{\pi a I/n}$, where $I = \sqrt{-1}$. Then, since $\{\omega^{-2j} : j = 0, \ldots, n-1\}$ are all the nth roots of unity, it follows that

$$\prod_{j=0}^{n-1} (z - \omega^{-2j}) = z^n - 1.$$