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f-factors, complete-factors, and component-deleted subgraphs

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ABSTRACT

In this paper, we consider the relationship between f-factors and component-deleted subgraphs of graphs. Let G be a graph. A factor F of G is a *complete-factor* if every component of F is complete. If F is a complete-factor of G, and C is a component of F, then $G - V(C)$ is a *component-deleted subgraph*. Let $c(G)$ denote the number of components of G. Let f be an integer-valued function defined on $V(G)$ with $\sum_{x \in V(G)} f(x)$ even. Enomoto and Tokuda [H. Enomoto, T. Tokuda, Complete-factors and f-factors, Discrete Math. 220 (2000) 239–242] proved that if F is a complete-factor of G with $c(F) \geq 2$, and $G - V(C)$ has an f-factor for each component C of F, then G has an f-factor. We extend their result, and show that it suffices to consider a complete-factor of $G - X$ for some specified $X \subset V(G)$ instead of G. Let F be a complete-factor of $G - X$ with $c(F) \geq 2$. If $G - V(C)$ has an f-factor for each component C of F, then G has an f-factor in each of the following cases: (1) $\sum_{x \in X} \deg_G(x) \leq c(F) - 1$; (2) $c(F)$ is even and $\sum_{x \in X} \deg_G(x) \leq c(F) + 1$; (3) G has no isolated vertices and $\sum_{x \in X} \deg_G(x) \leq c(F) + |X| - 2$; or (4) G has no isolated vertices, $c(F)$ is even and $\sum_{x \in X} \deg_G(x) \leq c(F) + |X| - 1$. We show that the results in this paper are sharp in some sense.

1. Introduction

In this paper, we consider finite undirected graphs that may have loops and multiple edges. Let G be a graph. For $x \in V(G)$, we denote by $\deg_G(x)$ the degree and by $N_G(x)$ the set of neighbors of x in G. For a subset S of $V(G)$, we denote by $G[S]$ the subgraph of G induced by S. For disjoint subsets S and T of $V(G)$, we denote by $e_G(S, T)$ the number of edges joining S and T. If S or T is a singleton set $\{x\}$, we write x instead of $\{x\}$. For example, we write $e_G(x, T)$ instead of $e_G(\{x\}, T)$. In particular, for $x, y \in V(G)$, $e_G(x, y)$ is the number of edges joining x and y. For a subset $T \subset V(G)$ and a component C of $G - T$, $e_G(T, V(C))$ is sometimes described as $e_G(T, C)$. We denote by $c(G)$ the number of components of G.

Let f be an integer-valued function defined on $V(G)$. A spanning subgraph F of G such that $\deg_F(x) = f(x)$ for each $x \in V(G)$ is an *f-factor* of G. When f is a constant function taking a value r, an f-factor is an *r-factor*. Note that if G has an f-factor, then $\sum_{x \in V(G)} f(x)$ is even. When no fear of confusion arises, we often identify an f-factor with its edge set. In other words, for a graph G, we say that a subset F of $E(G)$ is an f-factor if the spanning subgraph of G with edge set F is an f-factor. Let $\mathcal{F}$ be a set of graphs. If each component C of a spanning subgraph F of G is isomorphic to some member of $\mathcal{F}$, then F is an *$\mathcal{F}$-factor*. Let K_n denote the complete graph on n vertices. Let $\mathcal{K}$ be the family of all complete graphs. We call a $\mathcal{K}$-factor a *complete-factor*. We sometimes regard a complete-factor as the set of its components. In this paper, $\sum_{C \in F}$ means to take the sum over all the components C of F. For any function $f(x)$ and a subset S of $V(G)$, we define $f(S) = \sum_{x \in S} f(x)$. In particular, $\deg_G(S) = \sum_{x \in S} \deg_G(x)$.

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Katerinis [2] proved that for even integer r, if $G - x$ has an r-factor for each $x \in V(G)$, then G has an r-factor. Enomoto and Tokuda generalized this result to general f-factors of a graph G with a complete-factor F. They proved the following theorem.

Theorem A (*Enomoto and Tokuda [1]*)**.** *Let G be a graph and F be a complete-factor of G with $c(F) \geq 2$. If $f: V(G) \to \mathbf{Z}^+ \cup \{0\}$ satisfies $f(V(G)) \equiv 0 \pmod 2$ and $G - V(C)$ has an f-factor for each component C of F, then G has an f-factor.*

Immediately, we have the following corollary from Theorem A by setting $F \cong |V(G)|K_1$.

Corollary B. *Let G be a graph of order at least two. If $f: V(G) \to \mathbf{Z}^+ \cup \{0\}$ satisfies $f(V(G)) \equiv 0 \pmod 2$ and $G - x$ has an f-factor for each vertex $x \in G$, then G has an f-factor.*

In Corollary B, every vertex x is examined to satisfy the condition that $G - x$ has an f-factor. In [3], the present author showed that when $F \cong |V(G)|K_1$ as in Corollary B, we may reduce the number of components of F whose deletions we need to consider under certain circumstances. These results are contained in Theorems C and D below.

Theorem C (*Kimura [3]*)**.** *Let G be a graph, and let X be a subset of $V(G)$ with $|V(G) - X| \geq 2$ and $\deg_G(X) \leq 2|V(G) - X| - 1$. If $f: V(G) \to \mathbf{Z}^+ \cup \{0\}$ satisfies $f(V(G)) \equiv 0 \pmod 2$ and $G - x$ has an f-factor for each $x \in V(G) - X$, then G has an f-factor.*

Theorem D (*Kimura [3]*)**.** *Let G be a graph without isolated vertices, and let X be a subset of $V(G)$ with $|V(G) - X| \geq 2$ and $\deg_G(X) \leq 2|V(G) - X| + |X| - 3$. When $|X| = 1$, assume $\deg_G(X) \leq 2|V(G) - X| - 1$. If $f: V(G) \to \mathbf{Z}^+ \cup \{0\}$ satisfies $f(V(G)) \equiv 0 \pmod 2$ and $G - x$ has an f-factor for each $x \in V(G) - X$, then G has an f-factor.*

In the current paper our main result combines the general complete-factor of Theorem A with the notion of setting aside $X \subset V(G)$ which need not be examined in the component deleting process, as introduced in Theorems C and D. The result is Theorem 1.

Theorem 1. *Let G be a graph, let X be a subset of $V(G)$ and let F be a complete-factor of $G - X$ with $c(F) \geq 2$. If $f: V(G) \to \mathbf{Z}^+ \cup \{0\}$ satisfies $f(V(G)) \equiv 0 \pmod 2$ and $G - V(C)$ has an f-factor for each component C of F, then G has an f-factor in the following cases:*

(i) $\deg_G(X) \leq c(F) - 1$;
(ii) *$c(F)$ is even and* $\deg_G(X) \leq c(F) + 1$;
(iii) *G has no isolated vertices and* $\deg_G(X) \leq c(F) + |X| - 2$; *or*
(iv) *G has no isolated vertices, $c(F)$ is even, and* $\deg_G(X) \leq c(F) + |X| - 1$.

We note that when $X = \emptyset$, our theorem is equivalent to Theorem A. When $F \cong (|V(G)| - |X|)K_1$, comparing Theorem 1 with Theorems C and D, we find that the bounds in Theorems C and D are weaker. However, our main theorem is not restricted to the case that F has no edges. As such, Theorem 1 can be successfully applied in cases where the hypotheses of Theorems C and D fail to be satisfied.

The following example shows an instance where our main theorem allows us to conclude that G has an f-factor while Theorems C and D cannot be used to determine that G has an f-factor. Let us consider six complete graphs H_i with $1 \leq i \leq 6$. Assume $H_1 \cong K_1$ with $V(H_1) = \{v_{11}\}$, $H_j \cong K_2$ with $V(H_j) = \{v_{j1}, v_{j2}\}$ and $2 \leq j \leq 4$, and $H_k \cong K_3$ with $V(H_k) = \{v_{k1}, v_{k2}, v_{k3}\}$ and $5 \leq k \leq 6$. Let G be the graph defined by

$$V(G) = \bigcup_{1\leq i\leq 6} V(H_i),$$

$$E(G) = \bigcup_{1\leq i\leq 6} E(H_i) \bigcup_{5\leq k\leq 6} \{v_{11}v_{k1}, v_{11}v_{k2}, v_{k3}v_{31}, v_{k3}v_{41}\} \bigcup \{v_{21}v_{32}, v_{21}v_{42}, v_{22}v_{32}, v_{22}v_{42}, v_{32}v_{42}\}.$$

This graph is shown in Fig. 1. First, we use our Theorem 1 to show that G has an f-factor. Suppose $X = V(H_1)$. Let F be a complete-factor of $G - X$ such that H_i is a component of F with $2 \leq i \leq 6$. Now $\deg_G(X) = 4 = c(F) - 1 = c(F) + |X| - 2$. Define $f: V(G) \to \mathbf{Z}^+ \cup \{0\}$ by $f(x) = 2$ for every vertex $x \in V(G)$. Clearly, $f(V(G))$ is even. Now we can easily check that $G - V(H_i)$ has an f-factor, $2 \leq i \leq 6$. Thus G has an f-factor by our main theorem. Next, we show that we cannot determine whether G has an f-factor by Theorem C or D. Since both Theorems C and D require that $G - v$ has an f-factor for each $v \in V(G) - X$, we must have $v \in X$ for each $v \in V(G)$ with $G - v$ having no f-factor. It is easily checked that none of $G - v_{11}, G - v_{32}, G - v_{42}, G - v_{53}, G - v_{63}$ has an f-factor. So we can assume that $X' = \{v_{11}, v_{32}, v_{42}, v_{53}, v_{63}\} \subseteq X$ in Theorems C and D. Thus $\deg_G(X) \geq \deg_G(X') = 20 > 2|V(G) - X'| - 1 = 15 \geq 2|V(G) - X| - 1$ and $2|V(G) - X'| + |X'| - 3 = 18 \geq 2|V(G) - X| + |X| - 3$. Thus we cannot apply either Theorem C or D.

It should be noted that Theorem A cannot be used to determine that the graph G in this example has an f-factor. To see this we consider a complete-factor F of G. Since $G - v_{11}$ has no f-factor, if Theorem A is to be applied we must have the component C of F containing v_{11} non-trivial. Without loss of generality, C must also contain v_{51}. However, $G - \{v_{11}, v_{51}\}$ has no f-factor, so $V(C) = \{v_{11}, v_{51}, v_{52}\}$. Since $G - V(C)$ does have an f-factor, consider the component D of $F - V(C)$ containing v_{53}. Although $G - V(C)$ has an f-factor, we show that $G - V(D)$ has no f-factor. Since $G - v_{53}$ has no f-factor, D is not K_1. If $V(D) = \{v_{53}, v_{31}\}$, then $G - V(D)$ has no f-factor. Thus, we may assume $V(D) = \{v_{53}, v_{41}\}$. In this case also, it is easy to see that $G - V(D)$ has no f-factor.

Thus Theorem 1 provides a genuine extension of Theorem A and the possibility of guaranteeing the existence of f-factors in some cases where Theorems C and D fail.