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Hadwiger's Conjecture and inflations of the Petersen graph

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ABSTRACT

An inflation of a graph G is obtained by replacing vertices in G by disjoint cliques and adding all possible edges between any pair of cliques corresponding to adjacent vertices in G. We prove that the chromatic number of an arbitrary inflation F of the Petersen graph is equal to the chromatic number of some inflated 5-cycle contained in F. As a corollary, we find that Hadwiger's Conjecture holds for any inflation of the Petersen graph. This solves a problem posed by Bjarne Toft.

1. Introduction

Hadwiger's Conjecture [5], which is a fundamental conjecture in the theory of graph colouring, states that every k-chromatic graph contains a complete minor of order k. Hadwiger's Conjecture is a far-reaching generalisation of the Four Colour Problem, and it remains open for all k greater than 6. (See [12] for a survey on Hadwiger's Conjecture.)

A stronger conjecture, known to Dirac already in the early 1950s [3] (see also [12]), states that every k-chromatic graph contains a subdivision of the complete graph on k vertices. In 1979, Catlin [2] showed that this latter conjecture, which became known as *Hajós' Conjecture*, is false for all values of k greater than 6. Catlin's counterexamples are surprisingly simple; they are just inflations of the 5-cycle. Catlin's counterexamples to Hajós' Conjecture are not counterexamples to Hadwiger's Conjecture, but we do not know whether an inflation of some other small graph might yield a counterexample to Hadwiger's Conjecture. Of course, if there is a counterexample to Hadwiger's Conjecture, then it has to be found among the counterexamples to Hajós' Conjecture. Thomassen [11] proved that a graph G is perfect if and only if every inflation of G satisfies Hajós' Conjecture. This, in particular, means that any non-perfect graph can be inflated to a counterexample to Hajós' Conjecture.

Plummer et al. [10] proved that no counterexample to Hadwiger's Conjecture can be obtained by inflating a graph with independence number 2 and order at most 11. Bjarne Toft[1] posed the problem of proving that any inflation of the Petersen graph satisfies Hadwiger's Conjecture. The reason for considering the Petersen graph is the fact that it is a small and very symmetric graph which has turned out to be a counterexample to several previous conjectures in graph theory.

In this paper, we prove that the chromatic number of an arbitrary inflation G of the Petersen graph is equal to the chromatic number of some inflated 5-cycle contained in G. As a corollary, we obtain a positive solution to Toft's problem.

Notation and terminology. All graphs considered in this paper are finite, undirected, and without loops or multiple edges. Let G denote a graph. A *clique* of G is a complete subgraph of G, and a k-*clique* is a clique on exactly k vertices. The clique

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[1] Private communication from Bjarne Toft, *Kyoto RIMS Winter School of Graphs and Algorithms*, December 2008. Toft also presented the problem at the 21st *Cumberland Conference on Graph Theory, Combinatorics, and Computing*, May 2008 [13]. The problem was then passed on by André Kézdy at the *UofL Discrete Math Workshop* 2008 [6]. According to Toft, the problem dates back to Plummer, Stiebitz, and Toft's work on the paper *On a special case of Hadwiger's Conjecture* [10].

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number $\omega(G)$ is the largest k for which G contains a k-clique. Given a positive integer k, we let $[k]$ denote the set $\{1, \dots, k\}$. A *k-colouring* of G is a function φ from the vertex set $V(G)$ of G into the set $[k]$ such that $\varphi(u) \neq \varphi(v)$ for every edge uv from the edge set $E(G)$ of G; the smallest integer k for which G has a k-colouring is the *chromatic number* $\chi(G)$ of G. A graph is said to be *k-chromatic* if it has chromatic number equal to k. An *independent set* S of G is a subset of the vertex set of G with the property that no two vertices of S are joined by an edge in G; the maximum integer k for which there exists an independent set S of G of cardinality k is the *independence number* of G, and it is denoted $\alpha(G)$. We let $\mathcal{P}$ denote the Petersen graph.

Given some graph G of order n with vertices labelled $v_1, v_2, \dots, v_n$, and given non-negative integers $k_1, \dots, k_n$, we define the *inflation* $G(k_1, \dots, k_n)$ of G (with respect to this particular labelling and the integers $k_1, \dots, k_n$) to be the graph obtained from G by replacing each vertex v_i of G by a set V_i of k_i (fresh) vertices and adding edges such that V_i induces a k_i-clique of $G(k_1, \dots, k_n)$ and any two vertices u_i and u_j of distinct sets V_i and V_j are adjacent if the corresponding vertices v_i and v_j are adjacent in G.[2] If G is a graph, we shall often let G' and G'' denote inflations of G.

A graph F is a *minor* of a graph G if F can be obtained from G by deleting edges and/or vertices and contracting edges; a *complete minor of order k* of G is a minor of G which is a complete graph on k vertices. A subdivision of a graph is obtained by replacing its edges by paths so that the paths are pairwise internally disjoint. Concepts and notation used but not defined in this paper will be used as in [1].

We shall be using the following theorem of Lovász.

Theorem 1 (*Lovász [8]*)**.** *Any inflation of a perfect graph is a perfect graph.*

2. A few preliminary results

The Petersen graph is triangle-free but contains 12 distinct 5-cycles. First we need to understand how to optimally colour the vertices of any inflation of a 5-cycle. The following observation, which we need for later use, has a trivial proof.

Observation 2. *Let the vertices of the 5-cycle C_5 be labelled cyclically $v_1, \dots, v_5$, and let G' denote the inflation $C_5(k_1, \dots, k_5)$ of C_5 where each vertex v_i is inflated to a k_i-clique on a vertex set V_i of cardinality k_i. Suppose $\omega(G') \geq n(G')/2$ with $\omega(G') = k_1+k_2$. Then $k_5 \leq k_2$ and $k_3 \leq k_1$. Let A and B denote disjoint sets of cardinality k_1 and k_2, and let A' and B' denote fixed subsets of A and B of cardinality k_3 and k_5, respectively. Then there is an $\omega(G')$-colouring φ of G' with $\varphi(V_1) = A$, $\varphi(V_2) = B$, $\varphi(V_3) = A'$, $\varphi(V_5) = B'$, and $\varphi(V_4) \subseteq (A \setminus A') \cup (B \setminus B')$.*

The following result is a special case of a result previously obtained by Gallai (1969, unpublished; see [7]), by Fiorini and Wilson [4], and by Rothschild and Stemple (see [9, p. 241]).

Lemma 3. *The chromatic number of any inflation G' of the 5-cycle is equal to*

$$\max\left\{\omega(G'), \left\lceil \frac{n(G')}{2} \right\rceil\right\}. \tag{1}$$

Proof. Suppose that each vertex v_i of the 5-cycle $C_5 = v_1v_2v_3v_4v_5$ is inflated to a k_i-clique on a vertex set V_i of cardinality k_i, and let G' denote the resulting graph. We shall apply induction on the number of vertices of G'. If $n(G') \leq 1$, then (1) holds. Suppose $n(G') \geq 2$ and that the desired equality holds for any inflation G'' of C_5 with $n(G'') < n(G')$. Let φ denote a $\chi(G')$-colouring of G'. If no colour class of φ contains more than one vertex, then G' is a complete graph, and the desired result follows immediately. Thus, we may assume that some colour class of φ contains exactly two vertices, say u and v. We may, by symmetry, assume that $\omega(G')$ is equal to $k_1 + k_2$. If $\omega(G') \geq \lceil n(G')/2 \rceil$, then the desired result follows from Observation 2. Hence, we may assume $\omega(G') \leq \lceil n(G')/2 \rceil - 1 = \lceil (n(G') - 2)/2 \rceil$. The fact that φ is a $\chi(G')$-colouring of G' implies

$$\chi(G') = \chi(G' - u - v) + 1. \tag{2}$$

By the induction hypothesis, $\chi(G' - u - v)$ is equal to

$$\max\left\{\omega(G' - u - v), \left\lceil \frac{n(G') - 2}{2} \right\rceil\right\}. \tag{3}$$

Of course, $\omega(G' - u - v)$ is at most $\omega(G')$, and, as noted above, $\omega(G') \leq \lceil (n(G') - 2)/2 \rceil$. Hence, the maximum occurring in (3) is equal to

$$\left\lceil \frac{n(G') - 2}{2} \right\rceil.$$

This, along with (2), implies (1), and so the proof is complete. $\square$

[2] Thomassen [11] refers to inflated graphs as 'replication graphs'.