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Independent sets and non-augmentable paths in arc-locally in-semicomplete digraphs and quasi-arc-transitive digraphs ☆

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ABSTRACT

A digraph is arc-locally in-semicomplete if for any pair of adjacent vertices x, y, every in-neighbor of x and every in-neighbor of y either are adjacent or are the same vertex. A digraph is quasi-arc-transitive if for any arc xy, every in-neighbor of x and every out-neighbor of y either are adjacent or are the same vertex. Laborde, Payan and Xuong proposed the following conjecture: Every digraph has an independent set intersecting every non-augmentable path (in particular, every longest path). In this paper, we shall prove that this conjecture is true for arc-locally in-semicomplete digraphs and quasi-arc-transitive digraphs.

1. Introduction and terminology

We only consider finite digraphs without loops and multiple arcs. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any $x, y \in V(D)$, we will write $\overrightarrow{xy}$ or $x \to y$ if $xy \in A(D)$, and also, we will write $\overline{xy}$ if $\overrightarrow{xy}$ or $\overrightarrow{yx}$. For any $u, v, x, y \in V(D)$ if $\overline{uv}$, $\overrightarrow{xu}$ and $\overrightarrow{yv}$, then we will write $\overrightarrow{x}\,\overline{uv}\,\overleftarrow{y}$. For disjoint subsets X and Y of $V(D)$ or subdigraphs of D, $X \to Y$ means that every vertex of X dominates every vertex of Y, $X \Rightarrow Y$ means that there is no arc from Y to X and $X \mapsto Y$ means that both of $X \to Y$ and $X \Rightarrow Y$ hold. For a vertex x in D, its *out-neighborhood* $N^+(x) = \{y \in V(D) : xy \in A(D)\}$ and its *in-neighborhood* $N^-(x) = \{y \in V(D) : yx \in A(D)\}$. For a set $W \subseteq V$, let $N^+(W) = \cup_{w\in W} N^+(w) - W$, $N^-(W) = \cup_{w\in W} N^-(w) - W$. For a pair X, Y of vertex sets of D, define $[X, Y] = \{xy \in A(D) : x \in X, y \in Y\}$. Let D' be a subdigraph of D and $x \in V(D) - V(D')$. We say that x and D' are adjacent if x and some vertex of D' are adjacent. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. The *strong component digraph* $SC(D)$ of D is obtained by contracting strong components of D and deleting any parallel arcs obtained in this process. An *empty digraph* is a simple digraph in which no two vertices are adjacent.

A path $P = x_0x_1 \ldots x_k$ in D is *non-augmentable* if there exists no path $y_0y_1 \ldots y_s$ in $D - V(P)$ such that $\overrightarrow{y_sx_0}$ or $\overrightarrow{x_ky_0}$ or $\overrightarrow{x_{i-1}y_0}$ and $\overrightarrow{y_sx_i}$ for some $1 \le i \le k$. Clearly, a longest path must be a non-augmentable path, but the converse is not true. A path $Q = x_0x_1 \ldots x_t$ in D is *internally and initially non-augmentable* if there exists no path $y_0y_1 \ldots y_s$ in $D - V(Q)$ such that $\overrightarrow{y_sx_0}$ or $\overrightarrow{x_{i-1}y_0}$ and $\overrightarrow{y_sx_i}$ for some $1 \le i \le t$. A path P in D intersects a subset F of $V(D)$ if $V(P) \cap F \ne \emptyset$.

A digraph is *arc-locally in-semicomplete (arc-locally out-semicomplete)* if for any pair of adjacent vertices x, y, every in-neighbor (out-neighbor) of x and every in-neighbor (out-neighbor) of y either are adjacent or are the same vertex. A digraph is *quasi-arc-transitive* if for any arc xy, every in-neighbor of x and every out-neighbor of y either are adjacent or are the same

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vertex. A digraph is *quasi-antiarc-transitive* if for any arc xy, every in-neighbor of y and every out-neighbor of x either are adjacent or are the same vertex. For concepts not defined here we refer the reader to [1,2].

In [6], Laborde et al. conjectured that in every digraph, there exists an independent set intersecting every longest path and showed that this conjecture is true for symmetric digraphs. This conjecture is still open. Many classes of digraphs have kernels, such as transitive digraphs. In [4], Galeana-Sánchez and Gómez showed that this conjecture is true for digraphs having a kernel. In [5], Galeana-Sánchez and Rincón-Mejia proved the conjecture for line digraphs, arc-locally semicomplete digraphs, quasi-antiarc-transitive digraphs, quasi-transitive digraphs, path-mergeable digraphs, locally in-semicomplete digraphs, locally out-semicomplete digraphs, semicomplete digraphs and semicomplete k-partite digraphs, all of which are generalizations of tournaments except line digraphs. Note that arc-locally in-semicomplete digraphs, arc-locally out-semicomplete digraphs and quasi-arc-transitive digraphs are also generalizations of tournaments. In this paper, we will prove this conjecture for these three classes of digraphs.

2. Arc-locally in-semicomplete digraphs

Let us start with two classes of digraphs which are closely related to arc-locally in-semicomplete digraphs.

Let C be a cycle of length $k \geq 2$ and let $V_1, V_2, \ldots, V_k$ be pairwise disjoint vertex sets. *The extended cycle* $C[V_1, V_2, \ldots, V_k]$ is the digraph with vertex set $V_1 \cup V_2 \cup \cdots \cup V_k$ and arc set $\cup_{i=1}^{k}\{v_i v_{i+1} : v_i \in V_i, v_{i+1} \in V_{i+1}\}$, where subscripts are taken modulo k. That is, we have $V_1 \mapsto V_2 \mapsto \cdots \mapsto V_k \mapsto V_1$ and there are no other arcs in this extended cycle. Let H_1 and H_3 be two empty digraphs, H_2 be a trivial empty digraph, H_4 be a semicomplete digraph and let H be a digraph with vertex set $V(H_1) \cup V(H_2) \cup V(H_3) \cup V(H_4)$ and arc set $A(H_4) \cup \{uv : u \in V(H_3) \cup V(H_4), v \in V(H_1)\} \cup \{xy : x \in V(H_4), y \in V(H_3)\} \cup \{zw : z \in V(H_2), w \in V(H_3)\}$, where H_1, H_2, H_3 and H_4 are pairwise disjoint and one of $V(H_3)$ and $V(H_4)$ is permitted to be empty. Add some arcs between $V(H_2)$ and $V(H_1) \cup V(H_4)$ to H such that the resulting digraph D is strong and the vertex of H_2 is adjacent to every vertex of $H_1 \cup H_4$. It is easy to see that $H_1 \rightarrow H_2$, $H_2 \mapsto H_3$, $H_3 \cup H_4 \mapsto H_1$, $H_4 \mapsto H_3$. Such D is called a *T-digraph* with T-partition $(V(H_1), V(H_2), V(H_3), V(H_4))$.

The following result can be found in [7].

Lemma 2.1 (*[7]*)**.** *Let D be a strong arc-locally in-semicomplete digraph, then D is either a semicomplete digraph, a semicomplete bipartite digraph, an extended cycle or a T-digraph.*

The following lemmas play an important role in our paper.

Lemma 2.2. *Let D be an arc-locally in-semicomplete digraph and let D' be a non-trivial strong subdigraph of D. For any $s \in V(D) - V(D')$, if there exists a path from s to D', then s and D' are adjacent.*

Proof. Let $P = sx_1 \ldots x_k$ be a shortest path from s to D'. We prove that s is adjacent to some vertex in D' by induction on the length k of P. Obviously, the assertion holds when $k = 1$. For any $k \geq 2$, we suppose that the assertion holds for $k - 1$. Note that $x_1 \ldots x_k$ is a path of length $k - 1$. By the induction hypothesis, there exists a vertex $u \in V(D')$ such that u and x_1 are adjacent. Since D' is a non-trivial strong digraph, there exists a vertex $v \in V(D')$ such that $v \rightarrow u$. Then $\overline{sv}$ because $\overrightarrow{s}\,\overline{x_1u}\,\overleftarrow{v}$ and D is arc-locally in-semicomplete. The proof of Lemma 2.2 is complete. □

Lemma 2.3. *Let D' be a subdigraph of an arc-locally in-semicomplete digraph D and let $s \in V(D) - V(D')$ such that there exists an arc from s to D' and $s \Rightarrow D'$. Then each of the following holds:*

(a) *If D' is a path of even length and s dominates the terminal vertex of D', then s dominates the initial vertex of D'.*
(b) *If D' is a cycle and s dominates two consecutive vertices in D', then $s \mapsto D'$.*
(c) *If D' is an odd cycle, then $s \mapsto D'$.*

Proof. (a) Let $D' = x_0x_1 \ldots x_{2k}$. For $k = 0$ the assertion is trivial, so assume $k \geq 1$. By the hypothesis, $\overrightarrow{x_{2k-2}}\,\overline{x_{2k-1}x_{2k}}\,\overleftarrow{s}$ which implies that $\overline{sx_{2k-2}}$. Combining this with $s \Rightarrow D'$, we have $s \rightarrow x_{2k-2}$. Continuing in this way, it follows that $s \rightarrow x_0$.

(b) Let $D' = y_1y_2 \ldots y_ty_1$. Without loss of generality, assume that $s \rightarrow \{y_1, y_2\}$. Let $y \in V(D') - \{y_1, y_2\}$ be arbitrary. Note that one of the lengths of $D'[y, y_1]$ and $D'[y, y_2]$ must be even. By (a), $s \rightarrow y$. So $s \mapsto D'$ follows from $s \Rightarrow D'$.

(c) Let $D' = z_1z_2 \ldots z_{2k+1}z_1$. Without loss of generality, assume that $s \rightarrow z_{2k+1}$. Note that the length of $D'[z_1, z_{2k+1}]$ is even. By (a), we have that $s \rightarrow z_1$. Therefore, it follows from (b) that $s \mapsto D'$. The proof of Lemma 2.3 is complete. □

Lemma 2.4. *Let D be an arc-locally in-semicomplete digraph and let D' be a non-trivial strong induced subdigraph of D and let $s \in V(D) - V(D')$ such that there exists an arc from s to D' and $s \Rightarrow D'$. Then each of the following holds:*

(a) *If D' is a bipartite digraph with bipartition (X, Y) and s dominates a vertex of X, then $s \mapsto X$.*
(b) *If D' is a non-bipartite digraph, then $s \mapsto D'$.*