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Large cycles in 4-connected graphs

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ABSTRACT

The first result states that every 4-connected graph G with minimum degree δ and connectivity κ either contains a cycle of length at least $4\delta - \kappa - 4$ or every longest cycle in G is a dominating cycle. The second result states that any such graph under the additional condition $\delta \geq \alpha$ either contains a cycle of length at least $4\delta - \kappa - 4$ or is hamiltonian, where α denotes the independence number of G. Both results are sharp in all respects.

1. Introduction

Only finite undirected graphs without loops or multiple edges are considered. We reserve n, δ, κ and α for denoting the number of vertices (order), minimum degree, connectivity and independence number of a graph. A good reference for any undefined terms is [1]. A graph G is Hamiltonian if G contains a Hamilton cycle, i.e. a simple cycle of length n. A cycle C in G is a dominating cycle if every edge of G has a vertex in common with C. Further, a cycle C is said to be a D_3-cycle if every path of length at least 2 (having at least two edges) has a vertex in common with C.

In 1971, Nash-Williams [3] proved the first fundamental result concerning dominating cycles.

Theorem A ([3]). *Let G be a 2-connected graph and C a longest cycle in G. If $\delta \geq (n+2)/3$ then C is a dominating cycle.*

The reverse version of this theorem was established by Voss and Zuluaga [6].

Theorem B ([6]). *Let G be a 3-connected graph and C a longest cycle in G. Then either $|C| \geq 3\delta - 3$ or C is a dominating cycle.*

Nash-Williams [3] observed that the conclusion in Theorem A can be essentially improved under the additional condition $\delta \geq \alpha$.

Theorem C ([3]). *Every 2-connected graph with $\delta \geq \max\{(n+2)/3, \alpha\}$ has a Hamilton cycle.*

The reverse version of Theorem C easily follows from Theorem B.

Theorem D ([6]). *Let G be a 3-connected graph and C a longest cycle in G. If $\delta \geq \alpha$ then either $|C| \geq 3\delta - 3$ or C is a Hamilton cycle.*

The bounds in Theorems A and B can be essentially improved also without any essential limitations, namely by incorporating connectivity κ into these bounds.

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Theorem E ([5]). *Let G be a 3-connected graph and C a longest cycle in G. If $\delta \geq (n+2\kappa)/4$ then C is a dominating cycle.*

Theorem F ([4]). *Let G be a 4-connected graph and C a longest cycle in G. Then either $|C| \geq 4\delta - 2\kappa$ or G has a dominating cycle.*

Theorems E and F are sharp only for $\kappa = 3$ and $\kappa = 4$, respectively.

Recently, Yamashita (see [7], Corollary 8) lowered the minimum degree bound in Theorem E up to $(n+\kappa+3)/4$ without any additional limitations, providing a best possible result in all respects.

Theorem G ([7]). *Let G be a 3-connected graph and C a longest cycle in G. If $\delta \geq (n+\kappa+3)/4$ then C is a dominating cycle.*

In this paper we prove, in fact, the reverse version of Theorem G.

Theorem 1. *Let G be a 4-connected graph and C a longest cycle in G. Then either $|C| \geq 4\delta - \kappa - 4$ or C is a dominating cycle.*

To show that Theorem 1 is best possible in all respects, we need some examples of special graphs. Let a, b, t, k be integers with $k \leq t$ and let $H(a, b, t, k)$ denote the graph obtained from $tK_a + \overline{K}_t$ by taking any k vertices in subgraph $\overline{K}_t$ and joining each of them to all vertices of K_b.

The graph $4K_{\delta-2} + K_3$ shows that the connectivity condition $\kappa \geq 4$ in Theorem 1 cannot be relaxed by replacing it with $\kappa \geq 3$ when $\delta \geq 5$. The graph $H(2, \delta-\kappa+1, \delta-1, \kappa)$ shows that for each $\kappa \geq 4$, the conclusion $|C| \geq 4\delta - \kappa - 4$ cannot be strengthened by replacing it with $|C| \geq 4\delta - \kappa - 3$. Finally, the graph $H(1, 2, \kappa+1, \kappa)$ shows that the conclusion "is a dominating cycle" cannot be strengthened by replacing it with "is a Hamilton cycle". So, Theorem 1 is sharp in all respects.

The following theorem can be derived from Theorem G easily.

Theorem H ([7]). *Every 3-connected graph with $\delta \geq \max\{(n+\kappa+3)/4, \alpha\}$ has a Hamilton cycle.*

Similarly, the next theorem follows from Theorem 1.

Theorem 2. *Let G be a 4-connected graph and C a longest cycle in G. If $\delta \geq \alpha$ then either $|C| \geq 4\delta - \kappa - 4$ or C is a Hamilton cycle.*

The graphs $4K_{\delta-2} + K_3 (\delta \geq 5)$, $H(1, 2, \kappa+1, \kappa)$ and $H(2, n-3\delta+3, \delta-1, \kappa)$ show that the bounds in Theorem 2 are best possible.

In order to prove Theorem 1, we need the following result due to Jung [2] concerning D_3-cycles.

Theorem I ([2]). *Let G be a 4-connected graph and C a longest cycle in G. Then either $|C| \geq 4\delta - 8$ or C is a D_3-cycle.*

2. Notation and preliminaries

The set of vertices of a graph G is denoted by $V(G)$ and the set of edges by $E(G)$. For S a subset of $V(G)$, we denote by $G \setminus S$ the maximum subgraph of G with vertex set $V(G) \setminus S$. For a subgraph H of G we use $G \setminus H$ as short for $G \setminus V(H)$. The neighborhood of a vertex $x \in V(G)$ will be denoted by $N(x)$. Set $d(x) = |N(x)|$. For $X \subseteq V(G)$, we use $N(X)$ to denote $\cup_{x \in X} N(x) \setminus X$. Furthermore, for a subgraph H of G and $X \subseteq V(G)$, we define $N_H(X) = N(X) \cap V(H)$.

Paths and cycles in a graph G are considered as subgraphs of G. If Q is a path or a cycle, then the length of Q, denoted by $|Q|$, is $|E(Q)|$. We write a cycle C with a given orientation as $\overrightarrow{C}$. For $x, y \in V(C)$, we denote by $x\overrightarrow{C}y$, or sometimes by $C[x, y]$, the subpath of C in the chosen direction from x to y. For $x \in V(C)$, we denote the h-th successor and the h-th predecessor of x on $\overrightarrow{C}$ by x^{+h} and x^{-h}, respectively. We abbreviate x^{+1} and x^{-1} as x^+ and x^-, respectively. For $C[x^+, y^+]$ we also write $C(x, y)$. For each $X \subset V(C)$, we define $X^{+h} = \{x^{+h} | x \in X\}$ and $X^{-h} = \{x^{-h} | x \in X\}$.

Let G be an arbitrary graph, C a longest cycle in G and B a connected component of $G \setminus C$ with $V(B) = \{x_1, x_2\}$. Put

$$R = N_C(x_1) \cup N_C(x_2), \qquad M = N_C(x_1) \cap N_C(x_2), \qquad Y = R \cup R^+ \cup M^{+2},$$

$$A = R \setminus M, \qquad A_1 = N_C(x_1) \setminus M, \qquad A_2 = N_C(x_2) \setminus M.$$

The following statement follows immediately.

Claim 1. *Let G be a graph, C a longest cycle in G and B a connected component of $G \setminus C$ with $V(B) = \{x_1, x_2\}$. Then $d(x_i) = |A_i| + |M| + 1$ $(i = 1, 2)$.*

Since C is a longest cycle, the next four statements can be derived by standard arguments.

Claim 2. *Let G be a graph, C a longest cycle in G and B a connected component of $G \setminus C$ with $V(B) = \{x_1, x_2\}$. Then*

(1) $R \cap R^+ \cap M^{+2} = \emptyset$,
(2) $N(y) \cap (R^+ \cup M^{+2} \setminus \{y^+\}) = \emptyset$ *for each* $y \in R^+$,
(3) $N(y) \cap (R^+ \setminus \{y^-\}) = \emptyset$ *for each* $y \in M^{+2}$,
(4) $N(z) \cap (R^+ \cup M^{+2} \setminus \{y\}) = \emptyset$ *for each* $z \in N(y) \setminus V(C)$ *and* $y \in M^{+2}$.