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Extra two-fold Steiner pentagon systems

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ABSTRACT

A two-fold pentagon system is a decomposition of the complete 2-multigraph (every two distinct vertices joined by two edges) into pentagons. A two-fold Steiner pentagon system is a two-fold pentagon system such that every pair of distinct vertices is joined by a path of length two in exactly two pentagons of the system. We consider two-fold Steiner pentagon systems with an additional property : for any two vertices, the two paths of length two joining them are distinct. We determine completely the spectrum for such systems, and point out an application of such systems to certain 4-cycle systems.

1. Introduction

A *pentagon system* [a *two-fold pentagon system*, respectively] of order v is a pair $(V, \mathcal{P})$ where V is a v-set and $\mathcal{P}$ is a collection of pentagons with vertices in V such that every edge of the complete graph K_v [of the complete 2-multigraph $2K_v$, respectively] on V is contained in exactly one pentagon [in exactly two pentagons, respectively] of $\mathcal{P}$. It was shown by the second author in 1966 [9] that a pentagon system of order v exists if and only if $v \equiv 1$ or 5 (*mod* 10), and by the second author and Huang in 1975 [10] that a two-fold pentagon system exists if and only if $v \equiv 0$ or 1 (*mod* 5).

A *Steiner pentagon system* (SPS) [a *two-fold Steiner pentagon system* (TSPS), respectively] of order v is a pentagon system [a two-fold pentagon system, respectively] with the additional property that every pair of distinct vertices is joined by a path of length two in exactly one pentagon of the system [in exactly two pentagons of the system, respectively]. In 1984, the first author and Stinson [7] proved that an SPS of order v exists if and only if $v \equiv 1$ or 5 (*mod* 10), $v \neq 15$, and in 1986, the authors proved [6] that a TSPS of order v exists if and only if $v \equiv 0$ or 1 (*mod* 5).

Suppose $(V, \mathcal{P})$ is a TSPS(v), and suppose that for two vertices x, y the two paths of length two joining x and y are (x, a_{xy}, y) and (x, b_{xy}, y). Call a TSPS(v) *extra* if for any two distinct vertices $x, y \in V$, $a_{xy} \neq b_{xy}$, i.e., the two 'middle' vertices a_{xy}, b_{xy} are also distinct, in other words, the two 2-paths joining x, y in the pentagons of the TSPS must be distinct, so that (x, a_{xy}, y, b_{xy}) is a 4-cycle.

It is easily seen that not all TSPSs have this extra property. To start with, there exists no extra TSPS of order 5. At the same time, a TSPS of order 5 without the extra property exists: just take each of the two pentagons of the pentagon system of order 5 twice. We will often write ETSPS for extra TSPS.

Given an ETSPS(v), $(V, \mathcal{P})$, consider the collection of 4-cycles $\mathcal{C} = \{(x, a_{xy}, y, b_{xy}) : x, y \in V, x \neq y\}$. It is easily seen that $(V, \mathcal{C})$ is a 4-cycle system with $\lambda = 4$ (see Section 6 below for a proof). Since for $v \equiv 6, 10, 11, 15$ (*mod* 20), the index $\lambda = 4$ for which a 4-cycle system of order v exists, is the smallest possible, this provides both, a new proof of the existence of such 4-cycle systems, and a nice motivation for considering the extra property of two-fold Steiner pentagon systems. In Sections 2–5, we determine completely the spectrum for ETSPSs.

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2. Extra two-fold Steiner pentagon systems

We start with a simple lemma.

Lemma 2.1. *There exists no ETSPS(5).*

Proof. Let $V = \{1, 2, 3, 4, 5\}$ and let $(V, \mathscr{P}_5)$ be an ETSPS(5). The set $\mathscr{P}_5$ consists of four pentagons, and assume w.l.o.g that $(1, 2, 3, 4, 5) \in \mathscr{P}_5$. Since none of the paths $(1, 2, 3)$, $(2, 3, 4)$, $(3, 4, 5)$, $(4, 5, 1)$, $(5, 1, 2)$ can occur in any of the remaining three pentagons, this leaves only six candidates for inclusion into $\mathscr{P}_5$: (a) $(1, 2, 4, 5, 3)$, (b) $(1, 2, 5, 3, 4)$, (c) $(1, 3, 2, 5, 4)$, (d) $(1, 3, 5, 2, 4)$, (e) $(1, 3, 4, 2, 5)$, (f) $(1, 4, 2, 3, 5)$.

Only one of (a), (b) can be in $\mathscr{P}_5$ since both contain the edge $\{1, 2\}$, and similarly, only one of (e), (f) can be in $\mathscr{P}_5$ because of the edge $\{1, 5\}$. If both (c), (d) were in $\mathscr{P}_5$ then neither of (a), (b), (e), (f) could be in $\mathscr{P}_5$, thus only one of (c), (d) can be in $\mathscr{P}_5$. Both (a) and (d) contain the path $(1, 3, 5)$, so only one of (a), (d) can be in $\mathscr{P}_5$. Both (b) and (d) contain the path $(2, 5, 3)$, so only one of (b), (d) can be in $\mathscr{P}_5$. It follows that if (d) were in $\mathscr{P}_5$ then neither of (a), (b), (c) could be in $\mathscr{P}_5$, and only one of (e), (f) could be in $\mathscr{P}_5$, a contradiction. Thus (d) cannot be in $\mathscr{P}_5$. It follows that (c) *must* be in $\mathscr{P}_5$, thus neither (a) nor (f) can be in $\mathscr{P}_5$ since in that case the edge $\{4, 5\}$ and $\{2, 3\}$, respectively, would occur in three pentagons. Thus $\mathscr{P}_5$ consists of $(1, 2, 3, 4, 5)$, (b) $= (1, 2, 5, 3, 4)$, (c) $= (1, 3, 2, 5, 4)$, and (e) $(1, 3, 4, 2, 5)$, a final contradiction, since the edge $\{3, 4\}$ occurs now in three pentagons. □

Example 2.2. An ETSPS(6).

Consider $V = \{1, 2, 3, 4, 5, 6\}$, and the six pentagons $(1, 5, 3, 6, 4)$, $(1, 4, 5, 2, 6)$, $(1, 2, 3, 4, 6)$, $(1, 5, 4, 2, 3)$, $(1, 2, 6, 5, 3)$, $(2, 5, 6, 3, 4)$. The verification of the extra property is straightforward.

Remark. The above ETSPS(6) has the following curious property: if one of the pentagons contains the 2-path (a, b, c) then there are two other pentagons containing the 2-paths (b, a, c) and (b, c, a), respectively. If we take for every such triple of 2-paths (a, b, c), (b, a, c), (b, c, a) a block $\{a, b, c\}$, we obtain a BIBD $(6, 10, 5, 3, 2)$!

Example 2.3. An ETSPS(10).

Let $V = Z_9 \cup \{\infty\}$, and let $\mathscr{P}$ consist of pentagons $(i, i+1, i+7, i+5, i+4)$, $(\infty, i+3, i, i+4, i+2)$ $(mod\ 9)$.

For a verification of the extra property it suffices to check the 2-paths between the pairs x, y where either one of x, y equals ∞ or $\delta(x, y) \in \{1, 2, 3, 4\}$ (where $\delta(x, y) = min(|x-y|, 9-|x-y|)$).

vertices x, y	middle vertices a, b
∞, i	$i+3, i+7$
$i, i+1$	$\infty, i+6$
$i, i+2$	$i+3, i+4$
$i, i+3$	$i+1, i+8$
$i, i+4$	$i+6, i+8$. □

3. Supersimple $(v, 5, 2)$-BIBDs and ETSPS(v)s

A $(v, 5, 2)$-BIBD is a block design with v elements, with blocks of size 5 and $\lambda = 2$, i.e. each pair of distinct elements is contained in exactly two blocks. It was shown by Hanani [4] that a $(v, 5, 2)$-BIBD exists if and only if $v \equiv 1$ or 5 $(mod\ 10)$, $v \neq 15$. A $(v, 5, 2)$-BIBD is *simple* if it contains no repeated blocks, and is *supersimple* if any two blocks have at most two elements in common. A recent paper [1] has completed the determination of the spectrum for supersimple $(v, 5, 2)$-BIBDs.

Theorem 3.1 ([1]). *There exists a supersimple $(v, 5, 2)$-BIBD if and only if $v \equiv 1$ or 5 $(mod\ 10)$, $v \neq 5, 15$.*

The connection between supersimple $(v, 5, 2)$-BIBDs and ETSPS(v)s is given in the following theorem.

Theorem 3.2. *If there exists a supersimple $(v, 5, 2)$-BIBD then there exists an ETSPS(v).*

Proof. Let $(V, \mathscr{B})$ be a supersimple $(v, 5, 2)$-BIBD. Replace each block $B \in \mathscr{B}$ with a pentagon P on the vertices of B and its complementary pentagon $\bar{P}$. The resulting collection of pentagons $\mathscr{P}$ has the property that each pair of non-complementary pentagons have at most two vertices in common. Clearly, $(V, \mathscr{P})$ is a TPS(v). For a pair of vertices $x, y \in V$, let B', B'' be the two blocks of $\mathscr{B}$ containing both x and y. If $P, \bar{P}$ are the two pentagons obtained from B' then x, y are adjacent in one of $P, \bar{P}$, and are joined by a path of length two in the other of $P, \bar{P}$, and similarly for B''. Thus $(V, \mathscr{P})$ is an TSPS(v). Finally, assume that the extra property is violated by a pair of vertices, say u, v. That is, for the two 2-paths (u, a, v), (u, b, v) joining u and v, we have $a = b$. But since these two 2-paths arise from pentagons created from two distinct blocks B_1, B_2 of $\mathscr{B}$, the blocks B_1, B_2 have the three elements u, v, a in common, a contradiction with supersimplicity of $(V, \mathscr{B})$. □

Lemma 3.3. *There exists an ETSPS(15).*