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The vile, dopey, evil and odious game players

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ABSTRACT

Many of my friends celebrate anniversaries these days, be it 50th birthday, or 60th, 70th, 80th, even 90th. It is to be deplored that they all accumulate together, rather than distribute themselves uniformly over my lifetime! It is now Gert Sabidussi's turn, who just joined the octogenarians club, and it is a pleasure to dedicate a paper to him, since he did, and continues to do, excellent algebraic graph theory, including insights into the automorphism group of graphs, studies of stable graphs, Sabidussi representation theorems for symmetric graphs, Sabidussi's compatibility conjecture, Sabidussi graphs, etc. Many leading mathematicians throughout the world are working on problems and insights initiated by Gert. He does all this in a relaxed playful way, as I witnessed when I acquired my own Sabidussi number 1. Hence it is only natural to relate here the fun that Gert and his friend Wil had while playing games in Dubrovnik, where a grand conference took place in 2009 honoring Gert. Unfortunately, I had to skip that conference, but, unknown to them, I planted a listening device.

WIL: We have worked and published jointly on the complexity of graph focality. Now that one of us is already an octogenarian and the other is only a decade away, let us have some fun; let us play a game.

GERT: I'm game.

WIL: In the Fall 2009 issue of the MSRI gazette EMISSARY, Elwyn Berlekamp and Joe Buhler proposed the following puzzle: "Nathan and Peter are playing a game. Nathan always goes first. The players take turns changing a positive integer to a smaller one and then passing the smaller number back to their opponent. On each move, a player may either subtract one from the integer or halve it, rounding down if necessary. Thus, from 28 the legal moves are to 27 or to 14; from 27, the legal moves are to 26 or to 13. The game ends when the integer reaches 0. The player who makes the last move wins. For example, if the starting integer is 15, a legal sequence of moves might be to 7, then 6, then 3, then 2, then 1, and then to 0. (In this sample game one of the players could have played better!) Assuming both Nathan and Peter play according to the best possible strategy, who will win if the starting integer is 1000? 2000?"

Let us dub it the MARK game, since it is due to Mark Krusemeyer according to Berlekamp and Buhler.

GERT: To get a feel for the MARK game, I'd construct a small table listing its P-positions (Previous player wins) and N-positions (Next player wins). For example, $0 \in \mathcal{P}$, since the Next (first) player cannot move, so the Previous (second) player wins by default, $1 \in \mathcal{N}$ since Next can move to $0 \in \mathcal{P}$; and $2 \in \mathcal{P}$. In general, every position that has a follower in $\mathcal{P}$ is in $\mathcal{N}$, and every position all of whose followers are in $\mathcal{N}$ is in $\mathcal{P}$ ($\mathcal{P}$ and $\mathcal{N}$ are the set of all P- and N-positions respectively).

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WIL: (Extracting his palm computer)…Here it is!

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
P/N	P	N	P	N	N	N	P	N	P	N	P	N	N	N	P	N	N	N	P	N	N	N	P	N

GERT: It is hard to see what's going on…It might be useful to separate out the P-positions and the N-positions into two sequences.

WIL: Alright, the rearranged table below suggests that $b_n = 2a_n$ for every nonnegative integer n, $\mathcal{P} = \cup_{n\geq 0} b_n$, $\mathcal{N} = \cup_{n\geq 1} a_n$, where $N_n = a_n, n \geq 1$; $P_n = b_n, n \geq 0$. But what's N_n?…Oh I see, $N_n = \text{mex}\{P_i, N_i : 0 \leq i < n\}$ for every $n \geq 0$, where the mex of a finite subset of nonnegative integers is the least nonnegative integer not in the set. In particular, the mex of the empty set is 0. Notice that the sequences (for $n \geq 1$) are complementary: they split the positive integers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
a_n	0	1	3	4	5	7	9	11	12	13	15	16	17	19	20	21	23	25	27	28	29	31	33	35
b_n	0	2	6	8	10	14	18	22	24	26	30	32	34	38	40	42	46	50	54	56	58	62	66	70

n	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
a_n	36	37	39	41	43	44	45	47	48	49	51	52	53	55	57	59	60	61	63	64	65	67	68	69
b_n	72	74	78	82	86	88	90	94	96	98	102	104	106	110	114	118	120	122	126	128	130	134	136	138

n	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71
a_n	71	73	75	76	77	79	80	81	83	84	85	87	89	91	92	93	95	97	99	100	101	103	105	107
b_n	142	146	150	152	154	158	160	162	166	168	170	174	178	182	184	186	190	194	198	200	202	206	210	214

GERT: Since $15 \in \mathcal{N}$ which has the follower $14 \in \mathcal{P}$, we indeed see, as hinted by Berlekamp–Buhler, that Nathan could have played better by moving $15 \to 14$ rather than $15 \to 7$, thus securing his win! But for deciding 1000 and 2000, the above recursive computation of the P- and N-positions is not too convenient. Is there a "closed form" formula for them, I wonder?

WIL: Well, the second sequence is not a "spectrum", i.e., there exist no real α, γ such that $b_n = \cup_{n\geq 0}\lfloor n\alpha + \gamma\rfloor$ ($\lfloor x\rfloor$ is the integer part of the real number x), since a necessary — though not sufficient — condition for that is that for all $n \geq 0$, $b_{n+1}-b_n \in \{k, k+1\}$ for some integer k, and here the differences are 2 and 4. Since the sequences are complementary, also the first sequence is not a spectrum…However, the fact that there are two followers and one of them is halving, suggests to consider some sort of binary numeration system.

GERT: The simplest such system is the ordinary positional binary system…Indeed, it appears that $\mathcal{N}$ is the set of all vile numbers, and $\mathcal{P}$ is the set of all dopey numbers.

WIL: What are vile and dopey numbers?

GERT: The vile numbers are those whose binary representations end in an even number of 0s, and the dopey numbers are those that end in an odd number of 0s.

WIL: …No doubt their names are inspired by the evil and odious numbers, those that have an even and an odd number of 1's in their binary representation respectively. To indicate that we count 0s rather than 1s, and only at the tail end, the "ev" and "od" are reversed to "ve" and "do" in "vile" and "dopey". "Evil" and "odious" where coined by Elwyn Berlekamp, John Conway and Richard Guy while composing their famous book *Winning Ways*.

GERT: Precisely. Indeed, the sequence $\{a_n\}_{n\geq 1}$ consists of all alternately evil and odious numbers: a_{2n-1} odious, a_{2n} evil ($n \geq 1$); the same holds for $\{b_n\}_{n\geq 1}$, which is just a shift of $\{a_n\}_{n\geq 1}$…Talking about shifts, let $R_B(m)$ denote the representation