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How to play Reverse Hex

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ABSTRACT

We present new results on how to play Reverse Hex, also known as Rex, or Misère Hex, on $n \times n$ boards. We give new proofs – and strengthened versions – of Lagarias and Sleator's theorem (for $n \times n$ boards, each player can prolong the game until the board is full, so the first/second player can always win if n is even/odd) and Evans's theorem (for even n, the acute corner is a winning opening move for the first player). Also, for even $n \geq 4$, we find another first-player winning opening (adjacent to the acute corner, on the first player's side), and for odd $n \geq 3$, and for each first-player opening, we find second-player winning replies. Finally, in response to comments by Martin Gardner, for each $n \leq 5$, we give a simple winning strategy for the $n \times n$ board.

1. Background

Reverse Hex, also known as Rex or as Misère Hex, is the variant of Hex in which the player who connects his/her two sides loses.

Rex uses a Hex board, i.e., an $m \times n$ array of hexagons. There are two players: each is assigned his/her own color – say black or white – and two opposing sides of the board. Players alternate moves; white moves first. On each move, a player colors any empty cell. The loser is the player who forms a *connecting chain*—a set of cells of his/her color connecting his/her two opposing sides. See Fig. 1.

A Hex board with every cell colored has a connecting chain for exactly one player, so – like Hex – Rex cannot end in a draw. (For more on Hex, including a proof of this result, see the survey by the first author and Jack van Rijswijck [9] or the books by Cameron Browne [1,2]). Thus either the first player or the second player wins, i.e., has a winning strategy.

In the July 1957 *Mathematical Games* column in *Scientific American*, Martin Gardner introduced Hex and Rex to his readers. He reported a result by Robert Winder: for Rex on the $n \times n$ board, the first player wins if n is even, and the second player wins if n is odd [4–8]. Winder's proof was apparently never published, but in 1974 Ronald Evans proved that for even n the acute corner is a winning opening move for the first player [3], and in 1999 Jeffrey Lagarias and Daniel Sleator proved that each player can avoid creating a connecting chain until every cell is colored, which implies Winder's result [10].

In 1988, Martin Gardner commented further: "…*it is easy to see the win on the* 3×3 [board], [but the] 4×4 [board] *is so complex that a winning line of play for the first player remains unknown. David Silverman reported in a letter that he had found an unusual pairing strategy for a second-player win on the* 5×5". [6] (p. 183), [8] (p. 91).

In this paper, we say more about how to play Rex. We give new proofs and strengthened versions of Lagarias and Sleator's theorem and Evans's theorem. Also, for even $n \geq 4$, we find another first-player winning opening (adjacent to the acute corner, on the first player's side), and for odd $n \geq 3$, and for each first-player opening,we find a winning second-player reply. Finally, in response to Gardner's comments we give simple winning strategies for board sizes up to 5×5.

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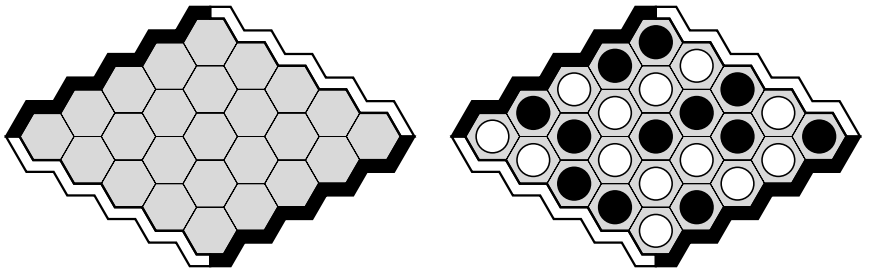

Fig. 1. An empty 5×5 board and a reverse Hex game lost by white.

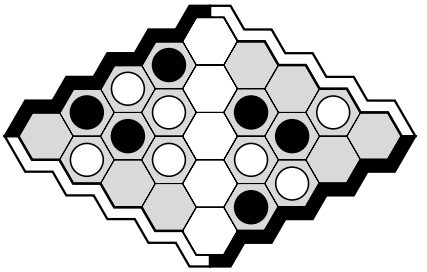

Fig. 2. A position with holes on the short diagonal. Neither player can win.

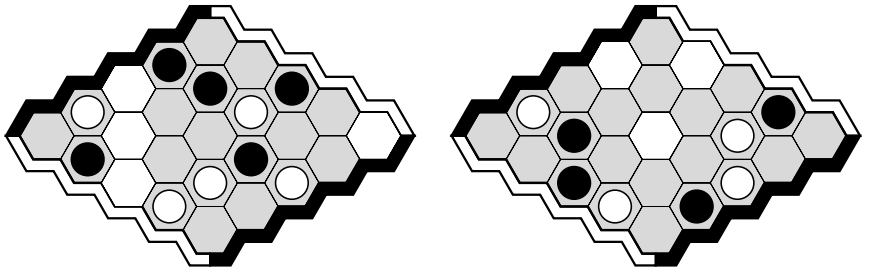

Fig. 3. Left, a color-symmetric position with symmetry axis the long diagonal. Right, a color-symmetric position with symmetry axis the short diagonal.

2. Notation, holes, and color symmetry

Throughout the paper, X is an arbitrary player and Y is X's opponent. We describe a game state (P, X) by giving the position P and the player to move X.

To simplify our arguments, we allow positions in which particular cells are excluded from play. We call such excluded cells – which neither player is allowed to color – *holes*. Thus, to describe a position, we specify for each cell whether it is black, white, a hole, or empty (uncolored). See Fig. 2.

Punctured Rex (PRex) is Rex played on a position which can have holes. A game of PRex can end with no connecting chain, in which case it is a draw.

Some PRex results which hold for an empty board hold more generally for positions which look the same for the two players. A position is *color symmetric* if the position is topologically identical for each player, i.e., if changing the color of each side and each colored cell and then either reflecting the board through the long diagonal, or reflecting the board through the short diagonal, yields a position that is identical to the original position; the associated diagonal is the *symmetry axis*. A color-symmetric position can have holes. Notice that every cell on the symmetry axis of a color-symmetric position must be either empty or a hole. See Fig. 3.

3. TRex

In reasoning about PRex, it is helpful to consider the variant we call Terminated PRex (TPRex), in which the game is terminated just before the board is filled. The rules for TPRex are as for PRex, with one change: *a player moves only if there remain at least two empty cells; if there remains at most one empty cell (and neither player has lost) then the game ends in a draw*. See Fig. 4.

The following lemma – inspired by an argument used by Lagarias and Sleator – shows that, for TPRex, (un)coloring a cell changes a winning state into (at worst) a non-losing state.