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Kotzig frames and circuit double covers

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ABSTRACT

A cubic graph H is called a *Kotzig graph* if H has a circuit double cover consisting of three Hamilton circuits. It was first proved by Goddyn that *if a cubic graph G contains a spanning subgraph H which is a subdivision of a Kotzig graph then G has a circuit double cover*. A spanning subgraph H of a cubic graph G is called a *Kotzig frame* if the contracted graph G/H is even and every non-circuit component of H is a subdivision of a Kotzig graph. It was conjectured by Häggkvist and Markström (*Kotzig Frame Conjecture, JCTB* 2006) that *if a cubic graph G contains a Kotzig frame, then G has a circuit double cover*. This conjecture was verified for some special cases: it is proved by Goddyn *if a Kotzig frame has only one component*, by Häggkvist and Markström (JCTB 2006) *if a Kotzig frame has at most one non-circuit component*. In this paper, the Kotzig Frame Conjecture is further verified for some families of cubic graphs with Kotzig frames H of the following types: (i) *a Kotzig frame H has at most two components*; or (ii) *the contracted graph G/H is a tree if parallel edges are identified as a single edge*. The first result strengthens the theorem by Goddyn. The second result is a further generalization of the first result, and is a partial result to the Kotzig Frame Conjecture for frames with multiple Kotzig components.

1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A circuit is a connected 2-regular graph. An even subgraph (or a cycle) is a graph such that the degree of each vertex is even. A bridge (or, cut-edge) of a graph G is an edge $e \in E(G)$ whose removal increases the number of components of G (that is, a bridge e is not contained in any circuit of G).

Graphs considered in this paper may contain loops or parallel edges. However, most of our graphs are bridgeless.

The following open problem has been recognized as one of the central problems in graph theory.

Conjecture 1.1 (*Circuit Double Cover Conjecture Szekeres [12], Seymour [11]*)**.** *Every bridgeless graph G has a family $\mathcal{C}$ of circuits such that every edge of G is contained in precisely two members of $\mathcal{C}$.*

Since an even subgraph is the union of a set of edge-disjoint circuits, the circuit double cover problem is equivalent to the even subgraph (cycle) double cover. An even subgraph double cover $\mathcal{F}$ of a graph G is called a k-even subgraph double cover (or k-cycle double cover) if $|\mathcal{F}| \le k$.

Comprehensive surveys about progress to this notoriously hard problem can be seen in papers [9] by Jaeger [8] by Jackson, etc., or the books [4,5,14].

In this paper, we study a special approach to the conjecture, which was initially started in [6], and, further generalized in [7].

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Definition 1.2. A cubic graph H is called a *Kotzig graph* if H has a 3-edge-coloring $c : E(H) \mapsto \{1, 2, 3\}$ such that $c^{-1}(i) \cup c^{-1}(j)$ is a Hamilton circuit of H for every pair $i, j \in \{1, 2, 3\}$. (That is, H is a *Kotzig graph* if it has a 3-circuit double cover.) The coloring c is called *a Kotzig coloring of* H.

Kotzig graphs were originally studied in [10]. Well-known examples of Kotzig graphs are $3K_2$, K_4, Möbius ladders M_{2k+1}, Heawood graph and dodecahedron.

Theorem 1.3 (*Goddyn [6], and also Häggkvist and Markström [7]*). *If a cubic graph G contains a spanning subgraph H which is a subdivision of a Kotzig graph, then G has a* 6*-even subgraph double cover.*

A 2-factor F of a cubic graph is even if every component of F is of even length. If a cubic graph G has an even 2-factor, then the graph G has many nice properties: *G is* 3*-edge-colorable*, *G has a circuit double cover, etc.* The following concepts were introduced in [7] as a generalization of even 2-factors.

Definition 1.4. Let G be a cubic graph. A spanning subgraph H of G is called a *frame* of G if the contracted graph G/H is an even graph.

Definition 1.5. Let G be a cubic graph. A frame H of G is called a *Kotzig frame* of G if, for each non-circuit component H_i of H, the suppressed graph $\overline{H_i}$ is a Kotzig graph.

The following conjecture was proposed in [7] as a generalization of Theorem 1.3.

Conjecture 1.6 (*Häggkvist and Markström [7]*). *Every cubic graph containing a Kotzig frame has a circuit double cover.*

In [7], Conjecture 1.6 was verified for some special cases.

Theorem 1.7 (*Häggkvist and Markström [7]*). *If a cubic graph G contains a Kotzig frame with at most one non-circuit component, then G has a* 6*-even subgraph double cover.*

In this paper, Conjecture 1.6 is further verified for some other families of cubic graphs, in which a Kotzig frame may contain more than one non-circuit components.

Theorem 1.8. *Let G be a cubic graph containing a Kotzig frame with at most two components. Then G has a* 6*-even subgraph double cover.*

Theorem 1.9. *Let G be a cubic graph and H be a Kotzig frame of G such that the contracted graph G/H is a tree if parallel edges are identified as a single edge. Then G has a* 6*-even subgraph double cover.*

2. Notation, terminology and basic lemmas

For most standardized notation and terminology, we follow [1–3,13], or [14].

Let G be a graph and H_1, H_2 be two vertex disjoint subgraphs of G. The set of edges with one end in H_1 and another in H_2 is denoted by $[H_1, H_2]$.

Let G be a graph. The suppressed graph of G, denoted by $\overline{G}$, is the graph obtained from G by replacing every maximal induced path by a single edge.

Definition 2.1. Let H be a bridgeless subgraph of a cubic graph G. A mapping $c : E(H) \to Z_3$ is called a *parity* 3*-edge-coloring* of H if, for each vertex $v \in H$ and each $\mu \in Z_3$,

$$|c^{-1}(\mu) \cap E(v)| \equiv |E(v) \cap E(H)| \pmod 2.$$

It is obvious that if H itself is cubic, then a parity 3-edge-coloring is a proper 3-edge-coloring (traditional definition).

The following lemma has been used frequently in many circuit covering problems.

Lemma 2.2. *If a cubic graph has an even* 2*-factor C, then G has a* 3*-even-subgraph double cover* $\mathcal{F}$ *such that* $C \in \mathcal{F}$.

3. Outline of the proofs

For a cubic graph G with a spanning Kotzig graph H (see Fig. 1), let $\{C_{12}, C_{01}, C_{02}\}$ be a circuit double cover of H consisting of three Hamilton circuits, and let $M = E(G) - E(H)$ (a matching). One can decompose M into three parts $\{M_{12}, M_{01}, M_{02}\}$ such that M_{ij} consists of chords of the circuit C_{ij}. Note that C_{ij} corresponds to a Hamilton circuit in the suppressed cubic graph $\overline{G_{ij}} = \overline{C_{ij} \cup M_{ij}}$. Therefore, by Lemma 2.2, let $\mathcal{F}_{ij}$ be a 3-even subgraph double cover of G_{ij} containing C_{ij}. So, $(\cup_{\{ij\}\subset\{0,1,2\}} \mathcal{F}_{ij}) - \{C_{12}, C_{02}, C_{01}\}$ is a 6-even subgraph double cover of G. (See Fig. 2.) This is the outline of the proof of Theorem 1.3 [6,7].