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Decomposition tree of a lexicographic product of binary structures

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ABSTRACT

Given a set S and a positive integer k, a binary structure is a function $B : (S \times S) \setminus \{(x, x);\ x \in S\} \longrightarrow \{1, \ldots, k\}$. The set S is denoted by $V(B)$ and the integer k is denoted by $\mathrm{rk}(B)$. With each subset X of $V(B)$ associate the binary substructure $B[X]$ of B induced by X defined by $B[X](x, y) = B(x, y)$ for any $x \neq y \in X$. A subset X of $V(B)$ is a clan of B if for any $x, y \in X$ and $v \in V(B) \setminus X$, $B(x, v) = B(y, v)$ and $B(v, x) = B(v, y)$. A subset X of $V(B)$ is a hyperclan of B if X is a clan of B satisfying: for every clan Y of B, if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$. With each binary structure B associate the family $\Pi(B)$ of the maximal proper and nonempty hyperclans under inclusion of B. The decomposition tree of a binary structure B is constituted by the hyperclans X of B such that $\Pi(B[X]) \neq \emptyset$ and by the elements of $\Pi(B[X])$. Given binary structures B and C such that $\mathrm{rk}(B) = \mathrm{rk}(C)$, the lexicographic product $B\lfloor C\rfloor$ of C by B is defined on $V(B) \times V(C)$ as follows. For any $(x, y) \neq (x', y') \in V(B) \times V(C)$, $B\lfloor C\rfloor((x, x'), (y, y')) = B(x, y)$ if $x \neq y$ and $B\lfloor C\rfloor((x, x'), (y, y')) = C(x', y')$ if $x = y$. The decomposition tree of the lexicographic product $B\lfloor C\rfloor$ is described from the decomposition trees of B and C.

1. Introduction

Given a set S and a positive integer k, a *binary structure* is a function $B : (S \times S) \setminus \{(x, x);\ x \in S\} \longrightarrow \{1, \ldots, k\}$. The set S is called the *vertex set* of B and is denoted by $V(B)$. The integer k is called the *rank* of B and is denoted by $\mathrm{rk}(B)$. The notion of binary structure extends the notions of graph, digraph and multigraph. For example, a graph $G = (V(G), E(G))$ is identified with the binary structure B_G of rank 2 defined on $V(B_G) = V(G)$ as follows. Given $x \neq y \in V(B_G)$, $B_G(x, y) = 1$ if $\{x, y\} \in E(G)$ and $B_G(x, y) = 2$ if $\{x, y\} \notin E(G)$. Given a binary structure B, associate with each subset X of $V(B)$ the binary *substructure* $B[X]$ of B induced by X defined by $B[X](x, y) = B(x, y)$ for any $x \neq y \in X$. Notice that $V(B[X]) = X$ and $\mathrm{rk}(B[X]) = \mathrm{rk}(B)$. For convenience, given $X \subsetneq V(B)$, $B[V(B) \setminus X]$ is also denoted by $B - X$ and by $B - x$ when $X = \{x\}$. With each binary structure B associate its *dual* $B^\star$ defined on $V(B^\star) = V(B)$ by $B^\star(x, y) = B(y, x)$ for $x \neq y \in V(B^\star)$. Notice that $\mathrm{rk}(B^\star) = \mathrm{rk}(B)$. Given binary structures B and C such that $\mathrm{rk}(B) = \mathrm{rk}(C)$, a bijection $f : V(B) \longrightarrow V(C)$ is an *isomorphism* from B onto C if $B(u, v) = C(f(u), f(v))$ for any $u \neq v \in V(B)$.

Given a binary structure B, a subset X of $V(B)$ is a *clan* [3] of B if for any $x, y \in X$ and $v \in V(B) \setminus X$, $B(x, v) = B(y, v)$ and $B(v, x) = B(v, y)$. For instance, $\emptyset$, $V(B)$ and $\{x\}$, $x \in V(B)$, are clans of B called *trivial* clans of B. Clearly B and $B^\star$ share the same clans. A clan of a graph is usually called a *module* [12]. A binary structure is *primitive* [3] if all its clans are trivial. A primitive graph is usually called *prime* [2]. Given a binary structure B, a subset X of $V(B)$ is a *hyperclan* (or a *prime* clan [3]) of B if X is a clan of B satisfying: for every clan Y of B, if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$. Notice that the trivial clans of B are hyperclans of B. With each binary structure B associate the family $\mathcal{H}(B)$ of the hyperclans of B. Then consider the family

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$\Pi(B)$ of the maximal elements of $\mathcal{H}(B) \setminus \{\emptyset, V(B)\}$ under inclusion. A hyperclan X of B is a *limit* of B if $\Pi(B[X]) = \emptyset$. We denote by $\mathcal{L}(B)$ the family of the limits of B. For example, notice that $\emptyset \in \mathcal{L}(B)$ and $\{x\} \in \mathcal{L}(B)$ for $x \in V(B)$.

Given a partial order O, a vertex x of O is *minimal* if there does not exist $y \in V(O)$ such that $y <_O x$. A partial order O is a *tree* if it is connected and if for each $v \in V(O)$, $O[\{v\} \cup \{u \in V(O) : v <_O u\}]$ is a linear order. With each binary structure B, associate the family

$$\mathcal{D}(B) = \bigcup_{X \in \mathcal{H}(B) \setminus \mathcal{L}(B)} \{X\} \cup \Pi(B[X]).$$

The set $\mathcal{D}(B)$ ordered by inclusion, denoted by $(\mathcal{D}(B), \subsetneq)$, is a tree classically called the *decomposition tree* of B [1,10,8]. As shown by the following example, the classic decomposition tree does not contain all the singletons. Let Λ be the usual linear order on the set $\mathbb{Z}$ of the integers. We consider the extension $\widehat{\Lambda}$ of Λ to $\mathbb{Z} \cup \{-\infty\}$ defined by $-\infty < n$ modulo $\widehat{\Lambda}$ for every $n \in \mathbb{Z}$. Then consider the graph G defined on $\mathbb{Z} \cup \{-\infty\}$ as follows. For any $u \neq v \in \mathbb{Z} \cup \{-\infty\}$, $\{u, v\} \in E(G)$ if $\max(u, v)$ is even. For every $n \in \mathbb{Z}$, set $n \downarrow = \{-\infty, n\} \cup \{m \in \mathbb{Z} : m < n\}$. We have $\mathcal{H}(G) \setminus \mathcal{L}(G) = \{n \downarrow : n \in \mathbb{Z}\}$ and $\Pi(G[n \downarrow]) = \{(n-1) \downarrow, \{n\}\}$ for each $n \in \mathbb{Z}$. Thus

$$\mathcal{D}(G) = \{n \downarrow : n \in \mathbb{Z}\} \cup \{\{n\} : n \in \mathbb{Z}\}.$$

Therefore $\{-\infty\} \notin \mathcal{D}(G)$. For convenience, we need all the singletons in the decomposition tree. Given a binary structure B, set

$$\widetilde{\mathcal{D}}(B) = \mathcal{D}(B) \cup \{\{x\} : x \in V(B)\}.$$

Clearly $(\widetilde{\mathcal{D}}(B), \subsetneq)$ is a tree as well. In what follows, it will be taken as *the* decomposition tree of B. Obviously $\widetilde{\mathcal{D}}(B) = \mathcal{D}(B)$ when B is finite.

Given two binary structures B and C such that $\mathrm{rk}(B) = \mathrm{rk}(C)$, the *lexicographic product* $B\lfloor C \rfloor$ of C by B is defined on $V(B\lfloor C \rfloor) = V(B) \times V(C)$ as follows. For any $(x, y) \neq (x', y') \in V(B) \times V(C)$,

$$B\lfloor C \rfloor((x, x'), (y, y')) = \begin{cases} B(x, y) & \text{if } x \neq y, \\ C(x', y') & \text{if } x = y. \end{cases}$$

Notice that $\mathrm{rk}(B\lfloor C \rfloor) = \mathrm{rk}(B) = \mathrm{rk}(C)$. Our purpose is to describe the decomposition tree of the lexicographic product $B\lfloor C \rfloor$ from the decomposition trees of B and of C. This should be useful to study the binary structures' idempotent under the lexicographic product, that is, the infinite binary structures B such that $B\lfloor B \rfloor$ is isomorphic to B. Sabidussi [11] introduced a construction to obtain graphs idempotent under the lexicographic product. We describe his construction as applied for binary structures. Consider a linear order L defined on a set $V(L)$ and a binary structure B with $|V(B)| \geq 2$. Choose a vertex of B and denote it by 0. Denote by ${}^{V(L)}V(B)$ the family of the functions $f : V(L) \longrightarrow V(B)$ such that $\{q \in V(L) : f(q) \neq 0\}$ is finite. The binary structure ${}^{L}B$ is defined on ${}^{V(L)}V(B)$ as follows: given $f \neq g \in {}^{V(L)}V(B)$, $({}^{L}B)(f, g) = B(f(s), g(s))$ where s denotes the smallest element of $\{q \in V(L) : f(q) \neq g(q)\}$ in the linear order L. Notice that $\mathrm{rk}({}^{L}B) = \mathrm{rk}(B)$. For a linear order L and a binary structure B, we obtain that $({}^{L}B)\lfloor {}^{L}B \rfloor$ is isomorphic to ${}^{2\lfloor L \rfloor}B$ where 2 denotes the usual linear order on $\{0, 1\}$. Consequently the binary structure ${}^{L}B$ is idempotent under the lexicographic product if $2\lfloor L \rfloor$ is isomorphic to L. For instance, consider the usual linear order on the set of rational numbers which is denoted by $\mathbb{Q}$ as well. We have $2\lfloor \mathbb{Q} \rfloor$ is isomorphic to $\mathbb{Q}$ and hence ${}^{\mathbb{Q}}B$ is idempotent under the lexicographic product for every binary structure B such that $|V(B)| \geq 2$. In fact, these binary structures are the only known binary structures idempotent under the lexicographic product. We hope that our structural study will permit a complete characterization of such binary structures. The decomposition tree of the graphs idempotent under the lexicographic product, obtained by the above construction, is entirely described in [5].

Let B and C be two binary structures such that $\mathrm{rk}(B) = \mathrm{rk}(C)$. The following two facts arise from our study. First, consider a clan W of $B\lfloor C \rfloor$. Although $\{x \in V(B) : \exists x' \in V(C),\ (x, x') \in W\}$ is a clan of B, $\{x' \in V(C) : \exists x \in V(B),\ (x, x') \in W\}$ is not always a clan of C. We characterize the clans W of $B\lfloor C \rfloor$ such that $\{x' \in V(C) : \exists x \in V(B),\ (x, x') \in W\}$ is not a clan of C (see Corollary 14). Second, consider $W \subseteq V(B) \times V(C)$ such that $|\{x \in V(B) : \exists x' \in V(C),\ (x, x') \in W\}| \geq 2$. We show that W is a hyperclan of $B\lfloor C \rfloor$ if and only if $W = \{x \in V(B) : \exists x' \in V(C),\ (x, x') \in W\} \times V(C)$ and $\{x \in V(B) : \exists x' \in V(C),\ (x, x') \in W\}$ is a hyperclan of B (see Lemma 15 and Proposition 16). On the other hand, $\{x\} \times V(C)$ is not always a hyperclan of $B\lfloor C \rfloor$ for $x \in V(B)$. The notion of a locally isolated vertex (see Section 3) allows us to analyze this situation (see Theorem 17). The second fact induces the main difficulty in decomposing $\widetilde{\mathcal{D}}(B\lfloor C \rfloor)$ into a lexicographical sum over $\widetilde{\mathcal{D}}(B)$ (see Eq. (1) in Section 6.2) which constitutes our principal result.

2. Connectivities and clan decomposition

2.1. Constant and linear binary structures

A binary structure B is $\{i\}$-*constant*, where $i \in \{1, \ldots, \mathrm{rk}(B)\}$, or simply *constant*, if $B(x, y) = i$ for any $x \neq y \in V(B)$.

Let B be a binary structure. Given $X \subsetneq V(B)$, $y \in V(B) \setminus X$ and $i \in \{1, \ldots, \mathrm{rk}(B)\}$, $B(y, X) = i$ means $B(y, x) = i$ for each $x \in X$. Similarly $B(X, y) = i$ means $B(x, y) = i$ for each $x \in X$. Given $X \subsetneq V(B)$ and $y \in V(B) \setminus X$, $y \sim X$ means that there are $i, j \in \{1, \ldots, \mathrm{rk}(B)\}$ such that $B(y, X) = i$ and $B(X, y) = j$. So a subset X of $V(B)$ is a clan of B if $y \sim X$ for each $y \in V(B) \setminus X$.