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On the distances between Latin squares and the smallest defining set size

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Abstract

We show that for each Latin square L of order $n \geq 2$, there exists a Latin square $L' \neq L$ of order n such that L and L' differ in at most $8\sqrt{n}$ cells. Equivalently, each Latin square of order n contains a Latin trade of size at most $8\sqrt{n}$. We also show that the size of the smallest defining set in a Latin square is $\Omega(n^{3/2})$.

Keywords: Latin square, Latin trade, defining set, critical set, Hamming distance.

1 Introduction

For each positive integer a, we use the notation $N(a)$ for the set of integers $\{0, 1, 2, \ldots, a-1\}$. A *partial Latin square* of order n is an $n \times n$ array, where each cell of the array is either empty or contains a symbol from $N(n)$, such that each symbol occurs at at most once per row and at most once per column. A *Latin square* is a partial Latin square in which no cell is empty.

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Indexing rows and columns by $N(n)$, we may consider a partial Latin square to also be a set of ordered triples of the form $(i, j, L(i, j))$. The *distance* (or *Hamming distance* [6]) between two partial Latin squares L and L' of the same order is then defined to be $|L \setminus L'|$. We show the following.

Theorem 1.1 *For each Latin square L of order n, there exists a Latin square $L' \neq L$ of order n such that $|L \setminus L'| \leq 8\sqrt{n}$.*

Theorem 1.1 is the first upper bound which is $o(n)$. This a step towards the possible truth of Conjecture 4.25 from [3]: For each Latin square L of order n, $\min\{|L \setminus L'| \mid L' \text{ is a Latin square of order } n \text{ and } L' \neq L\} = O(\log n)$.

We may also state Theorem 1.1 as a result about *Latin trades*. Given two distinct Latin squares L and L' of the same order n, $L \setminus L'$ is said to be a *Latin trade* with disjoint mate $L' \setminus L$. In terms of arrays, we say that two partial Latin squares are *row balanced* if corresponding rows contain the same set of symbols; *column balanced* is defined similarly. A Latin trade and its disjoint mate are thus a pair of partial Latin squares which occupy the same set of cells, are disjoint and are both row and column balanced. Theorem 1.1 thus implies:

Theorem 1.2 *Each Latin square L of order n contains a Latin trade T such that $|T| \leq 8\sqrt{n}$.*

As Latin squares are precisely operation tables for quasigroups, our main result can also be considered in the context of Hamming distances of algebraic objects (see [6] for more detail on this topic).

Let B_n be the Latin square formed by the addition table for the integers modulo n. The upper bound in the conjecture cited above cannot be decreased, since it is known that for each integer n, B_n contains a Latin trade of size $5\log_2 n$ [10].

If Cavenagh's Conjecture [3] is true, it may be that the back circulant Latin square is the "loneliest" of all Latin squares; i.e. the Latin square with greatest minimum distance to any other Latin square. It is shown in [9] that for any $\epsilon > 0$, almost all Latin squares of order n possess at least $O(n^{3/2-\epsilon})$ intercalates (of size 4). Thus we know that most Latin squares are not as "lonely" as the back circulant Latin square.

A *defining set* L' for a Latin square L of order n is a subset $L' \subseteq L$ such that if L'' is a Latin square of order n and $L' \subseteq L''$ then $L'' = L$. In other words, a defining set has unique completion to a Latin square of specified order. If T is a Latin trade in a Latin square L with disjoint mate T', $(L \setminus T) \cup T'$ is a Latin square distinct from L. Then it follows that if D is a defining set for a Latin