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Crowns in bipartite graphs

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Abstract

A set $S \subseteq V(G)$ is *stable* (or *independent*) if no two vertices from S are adjacent. Let $\Psi(G)$ be the family of all *local maximum stable sets* [6] of graph G, i.e., $S \in \Psi(G)$ if S is a maximum stable set of the subgraph induced by $S \cup N(S)$, where $N(S)$ is the neighborhood of S. If I is stable and there is a matching from $N(I)$ into I, then I is a *crown* of order $|I| + |N(I)|$, and we write $I \in Crown(G)$ [1].

In this paper we show that $Crown(G) \subseteq \Psi(G)$ holds for every graph, while $Crown(G) = \Psi(G)$ is true for bipartite and very well-covered graphs. For general graphs, it is **NP**-complete to decide if a graph has a crown of a given order [13]. We prove that in a bipartite graph G with a unique perfect matching, there exist crowns of every possible even order.

Keywords: maximum matching, bipartite graph, König-Egerváry graph, crown, order of a crown, local maximum stable set.

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1 Introduction

Throughout this paper G is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of G induced by X. The set $N_G(v)$ is the *neighborhood* of $v \in V(G)$, while $N_G[v] = N_G(v) \cup \{v\}$. If $|N_G(v)| = 1$, then v is a *leaf*, otherwise v is *internal*. The *neighborhood* $N_G(A)$ is $\{v \in V(G) : N_G(v) \cap A \neq \emptyset\}$, and $N_G[A] = N_G(A) \cup A$. If $A, B \subset V(G)$, then (A, B) denotes the set $\{ab : ab \in E(G), a \in A, b \in B\}$. A *matching* is a set of pairwise non-incident edges of G. The *matching number* $\mu(G)$ is the size of a *maximum matching* (a matching with the largest possible number of edges). A matching covering all the vertices is called perfect.

Proposition 1.1 *[6] Every tree contains a maximum matching covering all its internal vertices.*

If M is the unique perfect matching in the subgraph induced by the vertices that it saturates, then M is a *uniquely restricted matching* [5]. A stable set of maximum size, denoted $\alpha(G)$, is a *maximum* stable set, and by $\Omega(G)$ we mean the family of all maximum stable sets. Recall that if $\alpha(G) + \mu(G) = |V(G)|$, then G is a *König-Egerváry graph*. Each bipartite graph is König-Egerváry.

Theorem 1.2 *[10] G is a König-Egerváry graph if and only if each maximum matching of G is contained in $(S, V(G) - S)$, for every $S \in \Omega(G)$.*

A set $A \subseteq V(G)$ is *local maximum stable* in G if $A \in \Omega(G[N_G[A]])$ [6]. Let $\Psi(G)$ be the family of all local maximum stable sets of the graph G.

Theorem 1.3 *[11] Every $A \in \Psi(G)$ is a subset of some $S \in \Omega(G)$.*

Recall that G is a *well-covered* graph if all its maximal stable sets are of the same cardinality [12], and G is *very well-covered* if, in addition, it has no isolated vertices and $|V(G)| = 2\alpha(G)$ [4].

Theorem 1.4 *[8] If G is a very well-covered graph, then $G[N_G[S]]$ is a König-Egerváry graph, for every $S \in \Psi(G)$.*

If $S_j \in \Psi(G)$ for all $j \in \{1, ..., k\}$, and $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{k-1} \subset S_k$, then $\{S_j : 0 \leq j \leq k\}$ is called an accessibility chain for S_k.

Theorem 1.5 *[6] For trees, every $S \in \Psi(T)$ has an accessibility chain.*

If I is a stable set of G such that there exists a matching from $N_G(I)$ into I, then I is called a *crown* of G, and the number $|I| + |N_G(I)|$ is called the order of the crown I [1]. Let $Crown(G) = \{I : I \subseteq V(G) \text{ such that } I \text{ is a crown}\}$. It is clear that $\emptyset, \{v\} \in Crown(G)$, where v is an isolated vertex of G.