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Enumeration Problems on the Expansion of a Chord Diagram

Tomoki Nakamigawa [1,2]

Department of Information Science
Shonan Institute of Technology
1-1-25 Tsujido-Nishikaigan, Fujisawa,
Kanagawa 251-8511, Japan

Abstract

A chord diagram is a set of chords of a circle such that no pair of chords has a common endvertex. A pair of chords is called a crossing if the two chords intersect. A chord diagram E is called nonintersecting if E contains no crossing. For a chord diagram E having a crossing $S = \{x_1x_3, x_2x_4\}$, the expansion of E with respect to S is to replace E with $E_1 = (E \setminus S) \cup \{x_2x_3, x_4x_1\}$ or $E_2 = (E \setminus S) \cup \{x_1x_2, x_3x_4\}$. A chord diagram $E = E_1 \cup E_2$ is called complete bipartite of type (m, n), denoted by $C_{m,n}$, if (1) both E_1 and E_2 are nonintersecting, (2) for every pair $e_1 \in E_1$ and $e_2 \in E_2$, e_1 and e_2 are crossing, and (3) $|E_1| = m$, $|E_2| = n$. Let $f_{m,n}$ be the cardinality of the multiset of all nonintersecting chord diagrams generated from $C_{m,n}$ with a finite sequence of expansions. In this paper, it is shown $\sum_{m,n} f_{m,n}(x^m/m!)(y^n/n!)$ is $1/(\cosh x \cosh y - (\sinh x + \sinh y))$.

Keywords: enumeration, chord diagram, expansion, alternating permutation

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[2] Email:`nakami@info.shonan-it.ac.jp`

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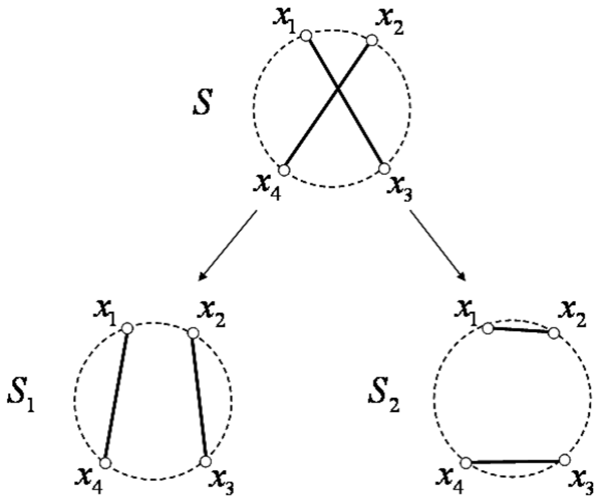

Fig. 1. The expansion of a chord diagram.

1 Introduction

Let us consider a set of chords of a circle. A set of chords is called a *chord diagram*, if they have no common endvertex. If a chord diagram consists of a set of n mutually crossing chords, it is called an *n-crossing*. A 2-crossing is simply called a crossing as well. If a chord diagram contains no crossing, it is called *nonintersecting*.

Let V be a set of $2n$ vertices on a circle, and let E be a chord diagram of order n, where each chord has endvertices of V. We denote the family of all such chord diagrams by $\mathcal{CD}(V)$. Let $x_1, x_2, x_3, x_4 \in V$ be placed on a circle in clockwise order. Let $E \in \mathcal{CD}(V)$. For a crossing $S = \{x_1x_3, x_2x_4\} \subset E$, let $S_1 = \{x_2x_3, x_4x_1\}$, and $S_2 = \{x_1x_2, x_3x_4\}$. The *expansion* of E with respect to S is defined as a replacement of E with $E_1 = (E \setminus S) \cup S_1$ or $E_2 = (E \setminus S) \cup S_2$ (see Figure 1).

For a chord diagram $E \in \mathcal{CD}(V)$ and a crossing $S \subset E$, we have two successors E_1 and E_2 of E. By iterating possible expansions, we have a multiset of nonintersecting chord diagrams. As is remarked in [1], the resulting multiset of nonintersecting chord diagrams generated from E with a set of expansions is uniquely determined.

Let us denote the multiset of nonintersecting chord diagrams generated by $E \in \mathcal{CD}(V)$ by $\mathcal{NCD}(E)$. For $E \in \mathcal{CD}(V)$, let us define $f(E)$ as the cardinality of $\mathcal{NCD}(E)$ as a multiset.

Let C_n be an n-crossing. In [1], the exponential generating function of