Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in DISCRETE MATHEMATICS
Electronic Notes in Discrete Mathematics 54 (2016) 73–78
www.elsevier.com/locate/endm

# Counting triangulations of balanced subdivisions of convex polygons [1]

Andrei Asinowski [2,3]

*Institute of Discrete Mathematics and Geometry*
*Vienna University of Technology*
*Vienna, Austria.*

Christain Krattenthaler [4,5]

*Faculty of Mathematics,*
*University of Vienna*
*Vienna, Austria.*

Toufik Mansour [6]

*Department of Mathematics,*
*University of Haifa*
*Haifa, Israel.*

**Abstract**

We compute the number of triangulations of a convex $k$-gon each of whose sides is subdivided by $r-1$ points. We find explicit formulas and generating functions, and we determine the asymptotic behaviour of these numbers as $k$ and/or $r$ tend to infinity. We connect these results with the question of finding the planar set of $n$ points in general position that has the minimum possible number of triangulations.

*Keywords:* Triangulations, generating functions, asymptotic analysis.

http://dx.doi.org/10.1016/j.endm.2016.09.014

# 1 Introduction

Let $k$ and $r$ be two natural numbers, $k \geq 3$, $r \geq 1$. Let $\mathrm{SC}(k,r)$ denote a convex $k$-gon in the plane each of whose sides is subdivided by $r-1$ points. (Thus, the whole configuration consists of $n := kr$ points.)

A *triangulation* of a planar point set S is a dissection of its convex hull by non-crossing diagonals into triangles. We denote the number of triangulations of $\mathrm{SC}(k,r)$ by $\mathsf{tr}(k,r)$. Triangulations of subdivided convex polygons were studied to some extent by Hurtado and Noy [4] [7] and by Bacher and Mouton [2]. We find enumeration formulas and precise asymptotic results for the numbers $\mathsf{tr}(k,r)$. Some of our results extend those from earlier papers, and answer questions and conjectures stated there and in the OEIS [5].

# 2 Formulas

The first step is developing an inclusion-exclusion formula for $\mathsf{tr}(k,r)$. [8]

**Theorem 2.1** *We have*

$$\mathsf{tr}(k,r) = \sum_{m=0}^{\lfloor r/2 \rfloor k} (-1)^m \, a_{k,r,m} \, C_{kr-m-2}, \tag{1}$$

*where $C_n$ is the nth Catalan number, and*

$$a_{k,r,m} := [x^m] \left( \sum_{\ell=0}^{\lfloor r/2 \rfloor} \binom{r-\ell}{\ell} x^\ell \right)^k .$$

**Proof (Sketch)** We construct a bijection between triangulations of $\mathrm{SC}(k,r)$ and a subset of triangulations of the convex $(kr)$-gon, determined by certain

---

[1] See `http://arxiv.org/abs/1604.02870` for a preprint of the full version of this paper.

[2] Research supported by the Austrian Science Foundation FWF, grant S50-N15, in the framework of the Special Research Program "Algorithmic and Enumerative Combinatorics".

[3] Email: `andrei.asinowski@tuwien.ac.at`.

[4] Research partially supported by the Austrian Science Foundation FWF, grant S50-N15, in the framework of the Special Research Program "Algorithmic and Enumerative Combinatorics".

[5] Email: `christian.krattenthaler@univie.ac.at`.

[6] Email: `toufik@math.haifa.ac.il`.

[7] Notice the difference in notation: our $k$ is their $r$, and our $r$ is their $k+1$.

[8] An equivalent formula was found in the earlier work [4] [2].