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# Structural and spectral properties of minimal strong digraphs

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**Abstract**

In this article, we focus on structural and spectral properties of minimal strong digraphs (MSDs). We carry out a comparative study of properties of MSDs versus trees. This analysis includes two new properties. The first one gives bounds on the coefficients of characteristic polynomials of trees (double directed trees), and conjectures the generalization of these bounds to MSDs. As a particular case, we prove that the independent coefficient of the characteristic polynomial of a tree or an MSD must be $-1$, 0 or 1. For trees, this fact means that a tree has at most one perfect matching; for MSDs, it means that an MSD has at most one covering by disjoint cycles. The property states that every MSD can be decomposed in a rooted spanning tree and a forest of reversed rooted trees, as factors. In our opinion, the analogies described suppose a significative change in the traditional point of view about this class of digraphs.

*Keywords:* Minimal strong digraphs, trees, characteristic polynomial, spanning tree.

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# 1 Introduction

A digraph is *strongly connected* or (simply) *strong* (SD) if every pair of vertices are joined by a path. An SD is *minimal* (MSD) if it loses the strong connection property when any of their arcs is suppressed. This class of digraphs has been considered under different points of view. See, for instance, [4,6].

We are also interested in the following nonnegative inverse eigenvalue problem [8]: given $k_1, k_2, \ldots, k_n$ real numbers, find necessary and sufficient conditions for the existence of a nonnegative matrix $A$ of order $n$ with characteristic polynomial $x^n + k_1x^{n-1} + k_2x^{n-2} + \cdots + k_n$. The coefficients of the characteristic polynomial are closely related to the cycle structure of the weighted digraph with adjacency matrix $A$ (see, for instance, [5]). The class of strong digraphs can easily be reduced to the class of minimal strong digraphs, so we are interested in any theoretical or constructive characterization of these classes of digraphs.

In [6], a sequentially generative procedure for the constructive characterization of the classes of MSDs is given. In addition, algorithms to compute unlabeled MSDs and their isospectral classes are described. These algorithms have been implemented to calculate the said classes of digraphs up to order 15 classified by their order and size [10]. We are also interested in properties regarding the spectral structure of this class of digraphs, mainly about the coefficients of the characteristic polynomial.

MSDs can be seen as a generalization of trees, as we pass from simple graphs to directed graphs. Although the structure of MSDs is much richer than that of trees, many analogies remain between the properties of both families. Other properties, nevertheless, undergo radical changes when passing from trees to MSDs.

In this article, we focus on structural and spectral properties of MSDs. We carry out a comparative study of properties of MSDs versus trees. An extended version of this work can be found in [7].

# 2 Minimal strong digraphs versus trees

In this paper we use some standard basic concepts and results about graphs. A *digraph* $D$ is a couple $D = (V, A)$, where $V$ is a finite nonempty set and

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