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Forcing clique immersions through chromatic number [1]

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Abstract

Building on recent work of Dvořák and Yepremyan, we show that every simple graph of minimum degree $7t + 7$ contains K_t as an immersion and that every graph with chromatic number at least $3.54t + 4$ contains K_t as an immersion.

Keywords: Graph immersion, Hadwiger conjecture, chromatic number.

1 Introduction

The graphs in this article are simple and finite. A fundamental question in

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graph theory is the relationship between the chromatic number of a graph G and the presence of certain structures in G. One of the most famous specific example of this type of question is Hadwiger's Conjecture [4] which states that for all positive integers t, every graph of chromatic number t contains K_t, the clique on t vertices, as a minor.

We consider a variant of Hadwiger's conjecture to *graph immersions* due to Lescure and Meynial [5] and, independently, to Abu-Khzam and Langston [1]. We first define graph immersions. Let G be a graph and $e, f \in E(G)$. Let x, y, and z be distinct vertices such that e has endpoints x and y and f has endpoints x and z. To *split off* the edges e and f, we delete e and f and add an edge e' with endpoints y and z. A graph G contains H as an *immersion* if a graph isomorphic to H can be obtained from a subgraph of G by repeatedly splitting off edges.

The conjecture explicitly states the following.

Conjecture 1.1 ([1], [5]) *For every positive integer t, every graph with no K_t immersion is properly $t-1$ colorable.*

One can immediately show that a minimum counterexample to Conjecture 1.1 has minimum degree $t-1$. Thus, the conjecture provides additional motivation for the natural question of what is the smallest mininum degree necessary to force a clique immersion. DeVos et al. [2] showed that minimum degree $200t$ suffices to force a K_t immersion in a graph, providing the first linear (in t) bound. This was recently improved by Dvořák and Yepremyn [3].

Theorem 1.2 *[3] Every graph with minimum degree at least $11t + 7$ contains an immersion[3] of K_t.*

We give a new result on clique immersions in dense graphs; we leave the exact statement for Section 2 below. As a consequence, it is possible to improve the analysis in [3] and obtain the following bound.

Theorem 1.3 *Every graph with minimum degree at least $7t + 7$ contains an immersion of K_t.*

Conjecture 1.1 can be relaxed to consider the following question.

Problem 1.4 *What is the smallest function f such that for all positive t and all graphs G with $\chi(G) \geq f(t)$, it holds that G contains K_t as an immersion.*

As observed above, a minimum counterexample to Conjecture 1.1 has minimum degree $t-1$. Thus by Theorem 1.3, we get that chromatic number at

[3] Theorem 1.2 also holds true for a more restricted tructure called *strong immersion*.