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Electronic Notes in DISCRETE MATHEMATICS

Electronic Notes in Discrete Mathematics 54 (2016) 133–138 www.elsevier.com/locate/endm

Enumerating lattice 3-polytopes

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Abstract

A lattice 3-polytope $P \subset \mathbb{R}^3$ is the convex hull of finitely many points from $\mathbb{Z}^3$. The size of P is the number of integer points it contains, and the width of P is the minimum, over all integer linear functionals f, of the length of the interval $f(P)$.

We present our results on a full enumeration of lattice 3-polytopes via their size and width: for any fixed $n \geq d+1$ there are infinitely many 3-polytopes of width one and size n, but they are easy to describe (they lie between two consecutive lattice planes). Those of width larger than one are finitely many for each size, and the full list of them can be obtained by one of two methods: (a) Most of them contain two proper subpolytopes of width larger than one, and thus can be obtained from the list of size $n-1$ using computer algorithms. (b) The rest have very precise structural properties that allow for a direct enumeration of them.

We have implemented the algorithms in MATLAB and run it until obtaining the following: There are 9, 76, 496, 2675, 11698, 45035 and 156464 lattice 3-polytopes of width larger than one and of sizes 5, 6, 7, 8, 9, 10 and 11, respectively.

Keywords: Lattice polytopes, lattice points, finiteness, lattice width.

[1] Supported by grants MTM2011-22792, MTM2014-54207-P (both authors) and BES-2012-058920 (M. Blanco) of the Spanish Ministry of Science, and by the Einstein Foundation Berlin (F. Santos). Email contact: `monica.blancogomez@unican.es`; `francisco.santos@unican.es`

http://dx.doi.org/10.1016/j.endm.2016.09.024
1571-0653/

1 Basic notions and goal

A lattice d-dimensional polytope (or d-polytope) P is the convex hull of a finite set of points from $\mathbb{Z}^d$ with $\mathrm{aff}(P) = \mathbb{R}^d$. The *size* of P is its number of lattice points, that is, the cardinality of the set $P \cap \mathbb{Z}^d$. The *width of P with respect to* a certain linear functional $f : \mathbb{R}^d \to \mathbb{R}$ is the length of the interval $f(P) \subset \mathbb{R}$. The minimum of those, among all non-constant integer linear functionals f, is the *width of P*. The width of P can also be interpreted as the minimum lattice distance between two lattice hyperplanes enclosing P.

A *unimodular transformation* is a linear integer map $t : \mathbb{R}^d \to \mathbb{R}^d$ that preserves the lattice $\mathbb{Z}^d$. That is, $t(x) = A \cdot x + b$ for every $x \in \mathbb{R}^d$, for some $A \in \mathbb{Z}^{d\times d}$, $\det(A) = \pm 1$ and $b \in \mathbb{Z}^d$. Two lattice d-polytopes P and Q are *equivalent* if there exists a unimodular transformation t such that $t(P) = Q$. We consider lattice d-polytopes up to equivalence. Size, volume, width, combinatorial type, etc., are all invariant under unimodular transformations.

We want to classify all lattice 3-polytopes of any given size:

$$\mathcal{P}_d(n) := \{\text{lattice } d\text{-polytopes of size } n \text{ modulo unim. equivalence}\}.$$

The first case, $n = 4$, corresponds to *empty tetrahedra*: tetrahedra in which the only lattice points are the four vertices. Their classification is classical:

Theorem 1.1 (Classification of empty tetrahedra, White 1964 [5])

$$\mathcal{P}_3(4) = \{T(p,q) \mid p, q \in \mathbb{Z},\ 0 < p \le q,\ \gcd(p,q) = 1\},$$

where $T(p,q) := \mathrm{conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\}$*. Moreover,* $T(p,q)$ *has volume* q *and is equivalent to* $T(p',q)$ *if and only if* $p' \equiv \pm p^{\pm 1} \mod q$*.*

In particular, all empty tetrahedra have width one. Moreover, for any $n > 4$ it is easy to construct infinitely many non-equivalent lattice 3-polytopes of size n and width one. But in [2] we showed that those of larger width are finitely many, for each given size. That is, if we let

$$\mathcal{P}^*_d(n) := \{P \in \mathcal{P}_d(n) \mid \mathrm{width}(P) > 1\}$$

we have that:

Theorem 1.2 (Blanco and Santos, [2, Corollary 5.1]) *For each* $n \ge 4$*,* $|\mathcal{P}^*_3(n)| < \infty$*.*

That allows for a full enumeration of 3-polytopes of width larger than one in terms of their size. In [2,3] we gave the full classification up to size six: