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New family of small regular graphs of girth 5

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Abstract

A (k, g)-cage is a k-regular graph of girth g of minimum order. In this work, we focus on girth $g = 5$, where cages are known only for degrees $k \leq 7$. When $k \geq 8$, except perhaps for $k = 57$, the order of a $(k, 5)$-cage is strictly greater

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than $1 + k^2$. Considering the relationship between finite geometries and graphs we establish upper constructive bounds that improve the best so far.

Keywords: Regular graph, cage, girth, amalgam.

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1 Introduction and Results

All the graphs considered are finite and simple. Let G be a graph with vertex set V and edge set E. The *girth* of a graph G is the length $g = g(G)$ of a shortest cycle. The degree of a vertex $v \in V$ is the number of vertices adjacent to v. A graph is called k-regular if all its vertices have the same degree k, and bi-regular or (k_1, k_2)-regular if all its vertices have either degree k_1 or k_2. A (k, g)-graph is a k-regular graph with girth g and a (k, g)-cage is a (k, g)-graph with the fewest possible number of vertices; the order of a (k, g)-cage is denoted by $n(k, g)$. Cages were introduced by Tutte [15] in 1947 and their existence was proved by Erdős and Sachs [6] in 1963 for any values of regularity and girth. The lower bound on the number of vertices of a (k, g)-graph is denoted by $n_0(k, g)$, and it is calculated using the distance partition with respect either a vertex (for odd g), or and edge (for even g):

$$n_0(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \cdots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases}$$

Obviously a graph that attains this lower bound is a (k, g)-cage. There has been intense work related with *The Cage Problem*, focussed on constructing smallest (k, g)-graphs (for a complete survey of this topic see [8]).

In particular, we are interested in the cage problem for $g = 5$. In this case $n_0(k, 5) = 1 + k^2$ and it is well known that this bound is attained for $k = 2, 3, 7$ and perhaps for $k = 57$ (see [4]). For $k = 4, 5, 6$, the known graphs of minimum order are cages (see [11,12,16]).

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