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The maximum diameter of pure simplicial complexes and pseudo-manifolds

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Abstract

We construct d-dimensional pure simplicial complexes and pseudo-manifolds (without boundary) with n vertices whose combinatorial diameter grows as $c_d n^{d1}$ for a constant c_d depending only on d, which is the maximum possible growth. Moreover, the constant c_d is optimal modulo a singly exponential factor in d. The pure simplicial complexes improve on a construction of the second author that achieved $c_d n^{2d/3}$. For pseudo-manifolds without boundary, as far as we know, no construction with diameter greater than n^2 was previously known.

Keywords: Simplicial complex, hypergraph, pseudo-manifold, diameter, Hirsch conjecture

A *pure simplicial complex of dimension* $d-1$ (or a $(d-1)$-*complex*, for short) is any family C of d-element subsets of a set V (typically, $V = [n] := \{1, \dots, n\}$). Elements of C are called *facets* of C and any subset of a facet is called a *face*. More precisely, a k-face is a face with $k+1$ elements. Faces of dimensions 0, 1, and $d-2$ are called, respectively, *vertices*, *edges* and *ridges* of C. We will always assume V to be finite and, without loss of generality, that V equals the set of vertices of C. Observe that a pure $(d-1)$-complex is the same as a *uniform hypergraph of rank* d. Its facets are called *hyperedges* in the hypergraph literature.

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The *adjacency graph* or *dual graph* of a pure simplicial complex C, denoted $\mathrm{G}(C)$, is the graph having as vertices the facets of C and as edges the pairs of facets $X, Y \in C$ that differ in a single element (that is, those that share a ridge). Complexes with a connected adjacency graph are called *strongly connected*. The *combinatorial diameter* of C is the diameter, in the graph theoretic sense, of $\mathrm{G}(C)$.

We are interested in how large can the diameter of a pure simplicial complex be in terms of its dimension and number of vertices. For this we set:

$$H_{\mathrm{s}}(n,d) := \quad \text{maximum diameter of pure strongly connected } (d-1)\text{-complexes with } n \text{ vertices.}$$

It is known that this can be exponential in d:

Theorem 1 (Santos [8, Corollary 2.12]) *In fixed dimension $d-1$:*

$$\Omega\left(n^{\frac{2d}{3}}\right) \leq H_s(n,d) \leq \frac{1}{d-1}\binom{n}{d-1} \simeq \frac{n^{d-1}}{d!}.$$

The upper bound is obtained by simply counting the possible number of ridges, while the lower bound comes from a construction using the *join* operation. Another construction showing $H_{\mathrm{s}}(n,d) \geq \Omega\left(n^{\frac{d}{4}}\right)$ is contained in [6, Thm. 4.4]. We show a simple and explicit construction giving:

Theorem 2 *For every $d, n_0 \in \mathbb{N}$, there is an $n \geq n_0$ such that:*

$$H_s(n,d) \geq \frac{n^{d-1}}{(d+2)^{d-1}} - 3.$$

The proof of this and our other results can be found in [2]. Observe that this matches the upper bound in Theorem 1, modulo a factor in $\Theta(d^{3/2}e^{-d})$, since $d! \simeq e^{-d}d^d\sqrt{2\pi d}$.

Remark 3 *Our proof of Theorem 2 uses an arithmetic construction valid only when the number n of vertices is of the form $q(d+2)$ for a sufficiently large prime power q. This is why we state it only for infinitely many values of n. However, every interval $[m, 2m]$ contains an n of that form, because there is a power of 2 between $m/(d+2)$ and $m/2(d+2)$). Hence, the theorem is also valid "for every sufficiently large $n \in \mathbb{N}$", modulo an extra factor of 2^{d-1} in the denominator.*

Motivation for this question and relatives of it comes from the Hirsch