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Chromatic index, treewidth and maximum degree

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Abstract

We conjecture that any graph G with treewidth k and maximum degree $\Delta(G) \geq k + \sqrt{k}$ satisfies $\chi'(G) = \Delta(G)$. In support of the conjecture we prove its fractional version.

Keywords: edge-colouring, fractional colouring, treewidth

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1 Introduction

The least number $\chi'(G)$ of colours necessary to properly colour the edges of a (simple) graph G is either the maximum degree $\Delta(G)$ or $\Delta(G)+1$. But to decide whether $\Delta(G)$ or $\Delta(G)+1$ colours are needed is a difficult algorithmic problem [6].

Often, graphs with a relatively simple structure can be edge-coloured with only $\Delta(G)$ colours. This is the case for bipartite graphs (König's theorem) and for cubic Hamiltonian graphs. Arguably, one measure of simplicity is *treewidth*, how closely a graph resembles a tree.

Zhou et al. [14] observed a consequence of Vizing's adjacency lemma: any graph with treewidth k and maximum degree at least $2k$ has chromatix index $\chi'(G) = \Delta(G)$. Is this tight? No, it turns out. Using two recent adjacency lemmas ([9,13]), we obtain:

Proposition 1.1 *For any graph G of treewidth $k \geq 4$ and maximum degree $\Delta(G) \geq 2k-1$ it holds that $\chi'(G) = \Delta(G)$.*

This immediately suggests the question: how much further can the maximum degree be lowered? We conjecture:

Conjecture 1.2 *Any graph of treewidth k and maximum degree $\Delta \geq k + \sqrt{k}$ has chromatic index Δ.*

The bound is close to best possible: we construct graphs with treewidth k, maximum degree $\Delta = k + \lfloor \sqrt{k} \rfloor < k + \sqrt{k}$, and chromatic index $\Delta + 1$ (for infinitely many k). For other values of k the conjecture (if true) might be off by 1 from the best bound on Δ. This is, for instance, the case for $k = 2$, where the conjecture is known to hold. Indeed, Juvan et al. [8] show that series-parallel graphs with maximum degree $\Delta \geq 3$ are even Δ-edge-choosable.

In support of the conjecture we prove its fractional version: [4]

Theorem 1.3 *Any simple graph of treewidth k and maximum degree $\Delta \geq k + \sqrt{k}$ has fractional chromatic index Δ.*

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[4] The fractional chromatic index of a graph G is defined as

$$\chi'_f(G) = \min \left\{ \sum_{M \in \mathcal{M}} \lambda_M \; : \; \lambda_M \in \mathbb{R}_+, \sum_{M \in \mathcal{M}} \lambda_M \mathbf{1}_M(e) = 1 \quad \forall e \in E(G) \right\},$$

where $\mathcal{M}$ denotes the collection of all matchings in G and $\mathbf{1}_M$ the characteristic vector of M. For more details on the fractional chromatic index, see for instance Scheinerman and Ullman [10].