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Relation between number of kernels (and generalizations) of a digraph and its partial line digraphs

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Abstract

Let $D = (V, A)$ be a digraph, an arc subset $A' \subseteq A$ and a surjective mapping $\phi : A \to A'$ such that the set of heads of A' is V and $\phi|A' = Id$ and for every vertex $j \in V$, $\phi(\omega^-(j)) \subset \omega^-(j) \cap A'$. The *partial line digraph* of D, $\mathcal{L}D$, is the digraph with vertex set $V(\mathcal{L}D) = A'$ and set of arcs $A(\mathcal{L}D) = \{(ij, \phi(j,k)) : (j,k) \in A\}$. In this paper we prove the following results: Let k, l be two natural numbers such that $1 \leq l \leq k$, and D a digraph with $\delta^-(D) \geq 1$. Then the number of (k,l)-kernels of D is less than or equal to the number of (k,l)-kernels of $\mathcal{L}D$. Moreover, if $l < k$ and the girth of D is at least $l+1$, then these two numbers are equal. The number of semikernels of D is equal to the number of semikernels of $\mathcal{L}D$. Also we introduce the concept of (k,l)-Grundy function as a generalization of the concept of Grundy

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function and we prove that the number of (k, l)-Grundy functions of D is equal to the number of (k, l)-Grundy functions of every partial line digraph $\mathcal{L}D$.

Keywords: Digraphs, in-domination, kernel, Grundy function.

1 Introduction

Throughout the paper, $D = (V, A)$ denotes a loopless digraph with set of vertices V and arc set A. Let $\omega^-(x)$ stand for the set of arcs having vertex x as their terminal vertex, and $\omega^+(x)$ stand for the set of arcs having vertex x as their initial vertex. Thus, the *in-degree* of x is $d^-(x) = |\omega^-(x)|$ and the *out-degree* of x is $d^+(x) = |\omega^+(x)|$. The *minimum in-degree* (*minimum out-degree*) of D is $\delta^-(D) = \min\{d_D^-(x) : x \in V\}$ ($\delta^+(D) = \min\{d_D^+(x) : x \in V\}$ respectively). Moreover, given a set $U \subseteq V$, $\omega^-(U) = \{(x, y) \in A : y \in U \text{ and } x \notin U\}$. The *distance* from x to y is denoted by $d_D(x, y)$ and it is defined to be the length of a shortest $x \to y$ directed path.

A set $K \subset V(D)$ is said to be a *kernel* if it is both independent (for every two vertices $x, y \in K$, $d_D(x, y) \geq 2$,) and absorbing (a vertex not in K has a successor in K). This concept was first introduced in [8] by Von Neumann and Morgensten in the context of Game Theory as a solution for cooperative n-player games. The concept of a kernel is important to the theory of digraphs because it arises naturally in applications such as Nim-type games, logic, and facility location, to name a few. Several authors have been investigating sufficient conditions for the existence of kernels in digraphs, for a comprehensive survey see for example [3], [5] and Chapter 15 of [7].

Let l, k be two integers such that $l \geq 1$ and $k \geq 2$. A (k, l)*-kernel of a digraph* D is a subset of vertices K which is both k*-independent* ($d_D(u, v) \geq k$ for all $u, v \in K$) and l*-absorbing* ($d_D(x, K) \leq l$ for all $x \in V \setminus K$). Thus every kernel is a $(2, 1)$-kernel and a quasikernel, introduced in [6], is a $(2, 2)$-kernel.

Grundy functions are very useful in the context of game theory and they are nearly related to kernels as a digraph with Grundy function has also a kernel.

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