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# Homomorphisms of Strongly Regular Graphs

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**Abstract**

We prove that if $G$ and $H$ are primitive strongly regular graphs with the same parameters and $\varphi$ is a homomorphism from $G$ to $H$, then $\varphi$ is either an isomorphism or a coloring (homomorphism to a complete subgraph). Moreover, any such coloring is optimal for $G$ and its image is a maximum clique of $H$. Therefore, the only endomorphisms of a primitive strongly regular graph are automorphisms or colorings. This confirms and strengthens a conjecture of Peter Cameron and Priscila Kazanidis that all strongly regular graphs are cores or have complete cores. The proof of the result is elementary, mainly relying on linear algebraic techniques.

*Keywords:* graph homomorphisms, strongly regular graphs, Lovász theta, cores.

## 1 Introduction

A *homomorphism* between two graphs $G$ and $H$ is a function $\varphi : V(G) \to V(H)$ such that $\varphi(u) \sim \varphi(v)$ whenever $u \sim v$, where '$\sim$' denotes adjacency. Whenever a homomorphism exists from $G$ to $H$, we write $G \to H$, and if

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both $G \to H$ and $H \to G$ then we say that $G$ and $H$ are *homomorphically equivalent*. Given a homomorphism $\varphi$ from $G$ to $H$, we will refer to the subgraph of $H$ induced by $\{\varphi(u) : u \in V(G)\}$ as the *image* of $\varphi$. It is easy to see that a $c$-coloring of a graph $G$ is equivalent to a homomorphism from $G$ to the complete graph on $c$ vertices, $K_c$. More generally, we will refer to any homomorphism whose image is a clique (complete subgraph) as a *coloring*.

A homomorphism from a graph $G$ to itself is called an endomorphism, and it is said to be *proper* if it is not an automorphism of $G$, or equivalently, its image is a proper subgraph of $G$. A graph with no proper endomorphisms is said to be a *core*, and these play a fundamental role in the theory of homomorphisms since every graph is homomorphically equivalent to a unique core. We refer to the unique core homomorphically equivalent to $G$ as *the core of* $G$. It is known [3], and not difficult to show, that the core of $G$ is isomorphic to any vertex minimal induced subgraph of $G$ to which $G$ admits an endomorphism.

If the core of a graph $G$ is a complete graph $K_c$, then $G$ must contain a clique of size $c$ and must also be $c$-colorable. Therefore, $\omega(G) = \chi(G) = c$, where $\omega(G)$ and $\chi(G)$ are the clique and chromatic numbers of $G$ respectively. Conversely, if $\omega(G) = \chi(G) = c$, then the core of $G$ is $K_c$. If a graph is either a core or has a complete graph as a core, then it is said to be *core-complete*. Many known results on cores are statements saying that all graphs in a certain class are core-complete [1,2,4], and often it remains difficult to determine whether a given graph in the class is a core or has a complete core.

For some classes of graphs, something stronger than core-completeness can be shown. A graph $G$ is a *pseudocore* if every proper endomorphism of $G$ is a coloring. It follows that any pseudocore is core-complete, but the converse does not hold (consider a complete multipartite graph). Similarly, it is easy to see that any core is a pseudocore, but the converse does not hold in this case either, e.g. the Cartesian product of two complete graphs of size at least three.

In this work, we will focus on homomorphisms and cores of strongly regular graphs. An $n$-vertex $k$-regular graph is said to be *strongly regular* with parameters $(n, k, \lambda, \mu)$ if every pair of adjacent vertices has $\lambda$ common neighbors, and every pair of distinct non-adjacent vertices has $\mu$ common neighbors. For short, we will call such a graph an $SRG(n, k, \lambda, \mu)$. A strongly regular graph is called *imprimitive* if either it or its complement is disconnected. In such a case, the graph or its complement is a disjoint union of equal sized complete graphs. Homomorphisms of these graphs are straightforward, and so we will only consider *primitive* strongly regular graphs here. Because of this, from now on when we consider a strongly regular graph, we will implicitly assume