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Electronic Notes in DISCRETE MATHEMATICS

Electronic Notes in Discrete Mathematics 54 (2016) 277–282 www.elsevier.com/locate/endm

A geometric approach to dense Cayley digraphs of finite Abelian groups [1]

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Abstract

We give a method for constructing infinite families of dense (or eventually likely dense) Cayley digraphs of finite Abelian groups. The diameter of the digraphs is obtained by means of the related *minimum distance diagrams*. A *dilating* technique for these diagrams, which can be used for any degree of the digraph, is applied to generate the digraphs of the family. Moreover, two infinite families of digraphs with distinguished metric properties will be given using these methods. The first family contains digraphs with asymptotically large ratio between the order and the diameter as the degree increases (moreover it is the first known asymptotically dense family). The second family, for fixed degree $d = 3$, contains digraphs with the current best known density.

Keywords: Cayley digraph, minimum distance diagram, dilation, diameter, density.

[1] Research supported by the "Ministerio de Educación y Ciencia" (Spain) with the European Regional Development Fund under project MTM2011-28800-C02-01

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http://dx.doi.org/10.1016/j.endm.2016.09.048
1571-0653/

1 Introduction

Given a finite Abelian group Γ of order $N = |\Gamma|$, consider a *generating* set $B = \{\gamma_1, \ldots, \gamma_d\}$ of Γ. The *Cayley digraph of* Γ *with respect to* B is denoted by $G = \mathrm{Cay}(\Gamma, B)$. The set of vertices of G is $V(G) = \Gamma$ and the set of arcs is $A(G) = \{(\alpha, \beta) : \beta - \alpha \in B\}$. These digraphs are *regular* and *vertex transitive*. The (out and in) *degree* and *diameter* of G are denoted by $d(G)$ and $k(G)$, respectively. We write $G \cong G'$ when G and G' are isomorphic digraphs.

Let $\mathrm{NA}_{d,k}$ (respectively, $\mathrm{NC}_{d,k}$) be the maximum number of vertices that a Cayley digraph of an Abelian group (respectively, of a cyclic group), with degree d and diameter k, can have. Let us denote $\mathrm{lb}(d, k)$ the *lower bound* for $\mathrm{NA}_{d,k}$. Then, from [2, Theorem 9.1] of Dougherty and Faber in 2004, it follows that

$$\mathrm{lb}(d,k) = \frac{c}{d(\ln d)^{1+\log_2 e}} \frac{k^d}{d!} + O(k^{d-1}) \leq \mathrm{NA}_{d,k} < \binom{k+d}{k}, \tag{1}$$

for some constant c. As far as we know, no constructions of Cayley digraphs G of order $N(G) \sim \mathrm{lb}(b, k)$ are known.

The *density* $\delta(G)$ of a digraph G is defined by $\delta(G) = N(G)/(k(G) + d)^d$. Let us denote by $\Delta_{d,k} = \max\{\delta(G) : d(G) = d, k(G) = k\}$ and $\Delta_d = \max\{\Delta_{d,k} : d(G) = d\}$. The only value of d for which Δ_d is known is 2, Forcade and Lamoreaux in 2000 proved that $\Delta_2 = \frac{1}{3}$ in [5, Section 4]. That density is attained by $G_t = \mathrm{Cay}(\mathbb{Z}_t \oplus \mathbb{Z}_{3t}, \{(1, -1), (0, 1)\}$, with $N(G_t) = 3t^2$ and $k(G_t) = 3t - 2$. No Cayley digraph of a cyclic group attains this density for $d = 2$. Although the value of Δ_3 is not known, Fiduccia, Forcade and Zito in 1998 [3, Corollary 3.6] proved that $\Delta_3 \leq \frac{3}{25} = 0.12$. Numerical evidences seem to label this value as too optimistic. The maximum density attained by known Cayley digraphs is $\delta_0 = 0.084$ and they have been found by computer search. These digraphs are $G_0 = \mathrm{Cay}(\mathbb{Z}_{84}, \{2, 9, 35\})$ and $G_1 = \mathrm{Cay}(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{168}, \{(1, 0, 2), (0, 0, 9), (0, 1, 35)\})$ in [3, Table 1] and $G_1' \cong G_1$, $G_2 = \mathrm{Cay}(\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{252}, \{(0, 0, 2), (0, 1, 9), (1, 0, 35)\})$ and $G_3 = \mathrm{Cay}(\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{336}, \{(0, 1, 2), (0, 0, 9), (1, 0, 35)\})$ in [2, Table 8.2].

Remark 1.1 A large value of the ratio $N(G)/k(G)$ does not guarantee a large density of G.

In this work we study some metric properties of Cayley digraphs which allow us to provide two outstanding infinite families of dense or eventually likely dense digraphs. The first one has a large ratio N/k as d increases. Finally, each member of the second family has density $\delta = \delta_0$.