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The group inverse of subdivision networks

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Abstract

In this paper, given a network and a subdivision of it, we show how the Group Inverse of the subdivision network can be related to the Group Inverse of initial given network. Our approach establishes a relationship between solutions of related Poisson problems on both networks and takes advantatge on the definition of the Group Inverse matrix.

Keywords: Network, Subdivision, Laplacian matrix, Group Inverse.

1 Introduction and preliminaries

As many recent papers are devoted to the study of different parameters of the subdivision graph that can be inferred from its group inverse (such as effective resistances and Kirchooff index, for instance), the evaluation of the group inverse of subdivision networks is of undoubted interest.

In this paper $\Gamma = (V, E, c)$ denotes a *network;* that is, a finite, with no loops, nor multiple edges, connected graph, with n vertices that we can label

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$V = \{1, 2, \dots, n\}$ and m edges in E, in which each edge $\{i, j\}$ has been assigned a weight or *conductance* $c_{ij} > 0$. It is $c_{ij} = c_{ji}$ as defined on edges. In addition, when $\{i, j\} \notin E$ we define $c_{ij} = 0$ and, in particular, $c_{ii} = 0$ for any i. We define the *(weighted) degree of* i as $k_i = \sum_{j=1}^{n} c_{ij}$.

The *combinatorial Laplacian* of Γ is the $n \times n$ matrix L whose entries are $L_{ij} = -c_{ij}$ for all $i \neq j$ and $L_{ii} = k_i$. Therefore, for each vector $u \in \mathbb{R}^n$ and for each $i = 1, \dots, n$

$$\Big[L(u)\Big]_i = k_i u_i - \sum_{j=1}^{n} c_{ij} u_j = \sum_{j=1}^{n} c_{ij}(u_i - u_j). \tag{1}$$

It is well-known that $Lu = 0$ iff $u = a\mathsf{1}$, $a \in \mathbb{R}$ and $\mathsf{1} \in \mathbb{R}^n$ the vector whose entries equal one. Moreover, given $f \in \mathbb{R}^n$, the Poisson problem, e.g. the linear system $Lu = f$, has solution iff $\langle f, \mathsf{1} \rangle = 0$. In this case two different solutions differ up to a constant. Therefore, there exists a unique orthogonal to $\mathsf{1}$ solution to every compatible linear system $Lu = f$ (*Fredholm's alternative*).

For a square matrix M, its *group inverse,* denoted as $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$ and $MX = XM$. It is very well known, see [3, and the references therein], that $M^{\#}$ exists if and only if $\operatorname{rank}(M) = \operatorname{rank}(M^2)$. As the combinatorial Laplacian of Γ matrix, L is a square, symmetric matrix that satisfies $\operatorname{rank}(L) = \operatorname{rank}(L^2)$, thus $L^{\#}$ exists.

In particular, if e_i denotes the i–th unit vector for each $i = 1, \dots, n$, the linear system

$$L(u) = \mathrm{e}_i - \frac{1}{n}\mathsf{1} \tag{2}$$

has a unique solution which is orthogonal to $\mathsf{1}$. This solution will be denoted by $L_i^{\#}$, and we use this set of orthogonal to $\mathsf{1}$ solutions, varying the i vertices in V, to define an $n \times n$ matrix $L^{\#}$, the group inverse of L.

2 A Poisson Problem on a Subdivision Network

The *subdivision graph* (V^S, E^S), of a given graph (V, E), finite, with no loops, with no multiple edges and connected, is the one obtained by inserting a new vertex in every edge of E so that it is replaced by two new edges, let's say $\{i, v_{ij}\}$ and $\{j, v_{ij}\}$, once v_{ij} is the new inserted vertex in between $i, j \in V$ such that $\{i, j\} \in E$. In this way we obtain the new graph (V^S, E^S) which is also finite, with no loops and no multiple edges, and connected. The set of vertices $V^S = V \cup V'$ is the union of $V = \{1, 2, \cdots n\}$, those in the former