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Combinatorial Recurrences and Linear Difference Equations

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Abstract

In this work we introduce the triangular arrays of depth greater than 1 given by linear recurrences, that generalize some well-known recurrences that appear in enumerative combinatorics. In particular, we focussed on triangular arrays of depth 2, since they are closely related to the solution of linear three–term recurrences. We show through some simple examples how these triangular arrays appear as essential components in the expression of some classical orthogonal polynomials and combinatorial numbers.

Keywords: Combinatorial identities, triangular matrices, finite difference equations, orthogonal polynomials.

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1 Introduction

In this work we are interested in obtaining the unique solution of the initial value problem

$$a_k z_{k+1} - b_k z_k + c_{k-1} z_{k-1} = 0,\ k \in \mathbb{N}^*,\ z_0 = 1,\ z_1 = q, \tag{1}$$

where $q \in \mathbb{R}^*$, $(a_k), (b_k)$ and (c_k) are sequences of real numbers such that $a_k, c_k \neq 0$ for all k. Since (z_k) does not depend on a_0 and b_0 we always assume that $b_0 = q$. In addition, we also assume the usual convention that empty sums and empty products are defined as 0 and 1, respectively.

As we will show, the solution of (1) can be expressed through double sequences determined by recurrences that are related with combinatorial recurrences. Specifically, given $n \in \mathbb{N}^*$ we call *triangular array of depth* n to any double sequence, indexed from 0, such that $t_{k,m} = 0$ when $0 \le k < nm$. When necessary, we also assume that $t_{k,m} = 0$ when k or m are negative integers. If we identify double sequences with infinite matrices, then the subspace of triangular arrays of depth n is identify with a subspace of lower triangular matrices with nm null entries in column m. We consider triangular arrays given by recurrence relations. Specifically, given $g = (g_{k,m}), h = (h_{k,m})$, for any $n \in \mathbb{N}^*$ we define the triangular array of depth n, $(t_{k,m})$, as $t_{0,0} = 1$ and

$$t_{k,m} = g_{k-1,m} t_{k-1,m} + h_{k-n,m-1} t_{k-n,m-1}, \quad k \ge nm; \quad m \ge 0, \quad k + m \ge 1. \tag{2}$$

The linear character of the above recurrence allows us to solve it by determining each *column* in terms of the preceding ones.

Theorem 1.1 *Given the double sequences $(g_{k,m}), (h_{k,m})$, for any $n \in \mathbb{N}^*$ the unique triangular array of depth n satisfying the recurrence (2) is*

$$t_{k,0} = \prod_{j=0}^{k-1} g_{j,0}, \quad t_{k,m} = \sum_{s=2m}^{k} h_{s-n,m-1} t_{s-n,m-1} \prod_{j=s}^{k-1} g_{j,m},\ k \ge nm,\ m \ge 1.$$

In particular, $t_{nm,m} = \prod_{j=0}^{m-1} h_{nj,j}$ *for any* $m \in \mathbb{N}$.

We denote by $\mathscr{G}_2(g,h)$ the unique triangular array of depth 2 determined by the recurrence (2) and moreover, the double sequences g and h are called its *generators*.

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