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On limits of sparse random graphs

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Abstract

We present a notion of convergence for sequences of finite graphs $\{G_n\}$ that can be seen as a generalization of the Benjamini-Schramm convergence notion for bounded degree graphs, regarding the distribution of r-neighbourhoods of the vertices, and the left-convergence notion for dense graphs, regarding, given any finite graph F, the limit of the probabilities that a random map from $V(F)$ to $V(G_n)$ is a graph homomorphism. Furthermore, this presented convergence notion allows us to define, for each $p(n)$ and with high probability, a limit for a sequence of Erdős-Renyi random graphs with $G_n \sim G(n, p(n))$.

Keywords: Limits of graphs, limits of random graphs, ordered sets, Hardy fields.

1 Introduction

In the past few years, many notions convergence for infinite sequences of finite graphs $\{G_n\}$ have been pushed forward. One of the main purposes of these is to sharpen the comprehension of large graphs (graphs with a large number of

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vertices) as well as to find a new way of codifying problems in graph theory (see, for instance, [6]).

Let $v(G)$ and $e(G)$ denote, respectively, the number of vertices and edges of G. Some notable examples of convergence notions are the following:

L1 L-convergence (see [6]). For each finite graph F, $\lim_{n\to\infty} \frac{\hom(F,G_n)}{v(G_n)^{v(F)}} = \alpha_F \in [0,1]$, where $\hom(F,G_n)$ denotes the number of maps from the vertices of F to the vertices of G_n that preserve edges (the number of graph homomorphisms from F to G_n).

L2 Structural limits, regarding the convergence of the probability that a first order formula is satisfied in G_n, when the vertices are assigned to the free variables of the formula uniformly and random. Different notions of convergence arise by considering different fragments of first order logic formulas, such as those formulas whose satisfiability depend, for some $r \in \mathbb{N}$, on the r-neighbourhood of the variables ($\mathrm{FO}_{\mathrm{local}}$-convergence, see [8]).

L3 BS-convergence [1]. A sequence $\{G_n\}$ of graphs with degree bounded by d is said to be BS-convergence if the probability distributions of the rooted r-neighbourhoods converge (in distribution) for every r. That is to say, the probability that an r-neighbourhood of the uniform and randomly chose v in G_n looks like a particular rooted graph converge when n goes to infinity. [3]

The notions above presented have certain restrictions on its applicability. For instance, the L-convergence trivializes for sequences of graphs where the number of edges is subquadratic. $\mathrm{FO}_{\mathrm{local}}$-convergence, although it can be applied to sequences of graphs regardless of the density, it becomes restrictive for dense sequences: the change in one edge for each graph may radically change the behaviour of the sequence. In other cases the restriction is placed beforehand. For instance in the BS-convergence, the sequences considered have a uniform bound on the maximum degree of the graphs.

Several convergence notions have been introduced to refine and extend the convergence notions L1 and L3. Some examples are [9], [2,3], [7]. However, these either cannot be applied to sequence of graphs for large densities of edges, such as [7], or they trivialize for very sparse graphs such as [9] and [2,3] for bounded-degree graph sequences.

A natural question is to consider the limit of a sequence of Erdős-Renyi

[3] Since the maximum degree is bounded, the number of possible r-neighbourhoods is also bounded.