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On Balanced Coloring Games in Random Graphs

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Abstract

Consider the balanced Ramsey game, in which a player has r colors and where in each round r random edges of an initially empty graph on n vertices are presented. The player has to immediately assign a different color to each edge and her goal is to avoid creating a monochromatic copy of some fixed graph F for as long as possible. The Achlioptas game is similar, but the player only loses when she creates a copy of F in one distinguished color. We show that there is an infinite family of non-forests F for which the balanced Ramsey game has a different threshold than the Achlioptas game, settling an open question by Krivelevich et al. We also consider the natural vertex analogues of both games and show that their thresholds coincide for all graphs F, in contrast to our results for the edge case.

Keywords: Coloring games, Ramsey theory, random graphs

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1 Introduction

1.1 The balanced Ramsey game

Consider the following probabilistic one-player game. Starting with the empty graph on n vertices, in each round r new edges are sampled uniformly at random from all non-edges and inserted into the graph. The player has r colors at her disposal and must color these r edges immediately subject to the restriction that each color is assigned to exactly one of the r edges. Her goal is to avoid creating a monochromatic copy of some fixed graph F for as long as possible. We refer to this game as the *balanced Ramsey game*; it was introduced by Marciniszyn et al. in [3].

The typical duration of this game when played with an optimal strategy is formalized by the notion of its *threshold function* $N_0(F, r, n)$. Specifically, we say that $N_0(F, r, n)$ is a threshold function for the game (for a fixed graph F and a fixed integer $r \geq 2$) if for any function $N(n) = o(N_0)$, Painter can a.a.s. [4] 'survive' for at least N rounds using an appropriate strategy, and if for any $N(n) = \omega(N_0)$, Painter a.a.s. cannot survive for more than N rounds regardless of her strategy. Note that this defines the threshold function only up to constant factors; therefore, whenever we compare two threshold functions and e.g. say that one is strictly higher than the other this refers to their orders of magnitude.

Standard arguments show that such a threshold function always exists for games of this type. The goal when studying these games usually is to determine their threshold function *explicitly*. For example, the threshold for the balanced Ramsey game where $F = K_3$ is a triangle and $r = 2$ was shown in [3] to be $N_0(K_3, 2, n) = n^{6/5}$.

More generally, the authors of [3] considered the balanced Ramsey game for $r = 2$ and analyzed a greedy strategy in which the player avoids coloring decisions that directly create a monochromatic copy of F, but plays arbitrarily otherwise. This yields a general lower bound on the threshold of the game. They also proved a matching upper bound (and thus determined the exact threshold function) for a restricted class of graphs F including cycles of arbitrary size. More recently, Prakash et al. [5] extended these results by considering the game where $r \geq 2$ is an arbitrary fixed integer, and an improved greedy strategy that focuses on an appropriately chosen subgraph $H \subseteq F$ instead of F itself. In particular, their analysis yields the first threshold results for the case where $F = K_\ell$ is a complete graph of size at least 4 (and r is large

[4] asymptotically almost surely, i.e. with probability tending to 1 as n tends to infinity