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# Minimum $C_k$-saturated graphs

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**Abstract**

A graph $G$ is called $H$-saturated if it does not contain any copy of $H$, but for any edge $e$ in the complement of $G$ the graph $G + e$ contains some $H$. The minimum size of an $n$-vertex $H$-saturated graph is denoted by $sat(n, H)$. We prove

$$sat(n, C_k) = n + n/k + O((n/k^2) + k^2)$$

holds for all $n \geq k \geq 3$, where $C_k$ is a cycle with length $k$.

We conjecture that our constructions are essentially optimal.

*Keywords:* graphs,cycles,extremal graphs, minimal saturated graphs.

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# 1 Introduction

A graph $G$ is said to be $H$-saturated if
— it does not contain $H$ as a subgraph, but
— the addition of any new edge (from $E(\overline{G})$) creates a copy of $H$.
Let $\mathrm{sat}(n, H)$ denote the *minimum* size of an $H$-saturated graph on $n$ vertices. Given $H$, it is difficult to determine $\mathrm{sat}(n, H)$ because this function is not necessarily monotone in $n$, or in $H$. Recent surveys are by J. Faudree, Gould, and Schmitt [11], and by Pikhurko [19] on the hypergraph case. It is known [17] that for every graph $H$ there exists a constant $c_H$ such that

$$\mathrm{sat}(n, H) < c_H n$$

holds for all $n$. However, it is not known if the $\lim_{n\to\infty} \mathrm{sat}(n, H)/n$ exists; Pikhurko [19] has an example of a four graph set $\mathcal{H}$ when $sat(n, \mathcal{H})/n$ oscillates, it does not tend to a limit.

Since the classical theorem of Erdős, Hajnal, and Moon [9] (they determined $\mathrm{sat}(n, K_p)$ for all $n$ and $p$), and its generalization for hypergraphs by Bollobás [5], there have been many interesting hypergraph results (e.g., Kalai [16], Frankl [14], Alon [1], using Lovász' algebraic method) but here we only discuss the graph case.

Remarkable asymptotics were given by Alon, Erdős, Holzman, and Krivelevich [2,10] (saturation and degrees). Bohman, Fonoberova, and Pikhurko [4] determined the sat-function asymptotically for a class of complete multipartite graphs. More recently, for multiple copies of $K_p$ Faudree, Ferrara, Gould, and Jacobson [12] determined $\mathrm{sat}(tK_p, n)$ for $n \geq n_0(p, t)$.

# 2 Cycle-saturated graphs

What is the saturation number for the cycle, $C_k$? This has been considered by various authors, however, in most cases it has remained unsolved. Here relatively tight bounds are given.

**Theorem 2.1** *For all $k \geq 7$ and $n \geq 2k - 5$*

$$\left(1 + \frac{1}{k+2}\right) n - 1 < \mathrm{sat}(n, C_k) < \left(1 + \frac{1}{k-4}\right) n + k - 42.$$

The case of $\mathrm{sat}(n, C_3) = n - 1$ is trivial; the cases $k = 4$ and $k = 5$ were established by Ollmann [18] in 1972 and by Ya-Chen [7] in 2009, resp.