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Linking rings structures and semisymmetric graphs: Cayley constructions

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ABSTRACT

An LR structure is a tetravalent vertex-transitive graph together with a special type of a decomposition of its edge-set into cycles. LR structures were introduced in Potočnik and Wilson (2014) as a tool to study tetravalent semisymmetric graphs of girth 4. In this paper, we consider algebraic methods of constructing LR structures, using number theory, Cayley graphs, affine groups, abelian groups and fields.

1. Introduction

This paper continues work from [13,14], where the present authors introduced *linking rings structures*, that is, certain decompositions of tetravalent vertex-transitive graphs into cycles (see Definition 1.1 for a precise definition). Our original motivation for studying such structures stems from the desire to understand tetravalent semisymmetric graphs. A regular graph is called *semisymmetric* provided that its automorphism group acts transitively on the edges but not on the vertices of the graph. Semisymmetric graphs were first studied by Folkman [5] and have become the focus of a rather lively research activity (see for example [2,3,7,10,16]).

It was proved in [13] that every tetravalent semisymmetric graph of girth 4 and order n arises either via the *partial line graph construction* from a linking rings structure of order $\frac{n}{2}$, or via the *subdivided*

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double construction from a tetravalent arc-transitive graph of order $\frac{n}{4}$ (see [13, Theorem 5.1] and Section 1.3 for details).

Recently, a new motivation for the study of linking rings has arisen. Encouraged by the successful compilation of complete lists of all cubic vertex-transitive graphs of order at most 1280 [11], all tetravalent arc-transitive graphs of order at most 640 and of all tetravalent half-arc-transitive graphs of order at most 1000 (see [12]), one feels tempted to attempt a construction of the list of all "small' tetravalent vertex-transitive graphs. As will be briefly discussed in Section 1.2, the graphs admitting linking rings structures form an important family of tetravalent vertex-transitive graphs, which will have to be studied in detail if one wants to succeed in this endeavor.

We shall continue this section with a brief summary of the definitions and results from [14]. In Sections 2–4, we present several constructions of linking rings structures, and in Section 5 we give a list of some "small" semisymmetric graphs arising from the linking rings structures presented in this paper. All of these semisymmetric graphs are new and have not been encountered previously in the literature.

1.1. Cycle decompositions and linking rings structures

Let Λ be a tetravalent graph and $\mathcal{C}$ a partition of $E(\Lambda)$ into cycles. We shall call such a pair $(\Lambda, \mathcal{C})$ a *cycle decomposition*.

Two edges of Λ will be called *opposite at vertex* v, if they are both incident with v and belong to the same element of $\mathcal{C}$. The *partial line graph* of a cycle decomposition $(\Lambda, \mathcal{C})$ is the graph $\mathbb{P}(\Lambda, \mathcal{C})$ whose vertices are edges of Λ, and two edges of Λ are adjacent as vertices in $\mathbb{P}(\Lambda, \mathcal{C})$ whenever they share a vertex in Λ and are not opposite at that vertex. Because the two edges at v that belong to one cycle are connected to both of the edges in the other cycle containing v, the edges at each vertex of Λ form a 4-cycle in $\mathbb{P}(\Lambda, \mathcal{C})$. Thus, the girth of $\mathbb{P}(\Lambda, \mathcal{C})$ is usually 4 and never any larger.

A cycle decomposition $(\Lambda, \mathcal{C})$ is said to be *flexible* provided that for every vertex v and each edge e containing v, there is a symmetry which fixes each point on the cycle $D \in \mathcal{C}$ containing e and interchanges the other two neighbors of v. The edges joining v to those neighbors are in some other cycle C of $\mathcal{C}$, and the symmetry is called a *C-swapper at v*.

A cycle decomposition $(\Lambda, \mathcal{C})$ is called *bipartite* if $\mathcal{C}$ can be partitioned into two subsets $\mathcal{G}$ and $\mathcal{R}$ so that each vertex of Λ meets one cycle from $\mathcal{G}$ and one from $\mathcal{R}$. Especially in constructions, we will refer to the edges of the cycles in $\mathcal{G}$ and those in $\mathcal{R}$ as *green* and *red*, respectively. Similarly, a swapper at v swapping the red (respectively, green) neighbors of v will be called a *red swapper* (respectively, *green swapper*).

The largest subgroup of $\mathrm{Aut}(\Lambda, \mathcal{C})$ preserving each of the sets $\mathcal{G}$ and $\mathcal{R}$ will be denoted by $\mathrm{Aut}^+(\Lambda, \mathcal{C})$, and we will think of it as the color-preserving group of $(\Lambda, \mathcal{C})$.

Definition 1.1. Let Λ be a connected tetravalent graph. A cycle decomposition $(\Lambda, \mathcal{C})$ is called a *linking rings structure* (or briefly, an *LR structure*) provided that it is bipartite, flexible and $\mathrm{Aut}^+(\Lambda, \mathcal{C})$ is transitive on vertices.

Note that the index of $\mathrm{Aut}^+(\Lambda, \mathcal{C})$ in $\mathrm{Aut}(\Lambda, \mathcal{C})$ is at most 2. If there is a symmetry of Λ which preserves $\mathcal{C}$ but interchanges the edge color sets $\mathcal{G}$ and $\mathcal{R}$ (that is, if $\mathrm{Aut}^+(\Lambda, \mathcal{C}) \neq \mathrm{Aut}(\Lambda, \mathcal{C})$), then we say that $(\Lambda, \mathcal{C})$ is *self-dual*.

It is easy to see that the color preserving group $\mathrm{Aut}^+(\Lambda, \mathcal{C})$ of an LR structure $(\Lambda, \mathcal{C})$ is transitive on $\mathcal{G}$ and on $\mathcal{R}$, implying that all cycles in $\mathcal{R}$ must have the same length, say p, and all cycles in $\mathcal{G}$ must be of the same length q. We then say that the LR structure $(\Lambda, \mathcal{C})$ is of *type* $\{p, q\}$. For a self-dual structure, of course, $p = q$.

1.2. Linking rings structures and tetravalent vertex-transitive graphs

A very natural approach to the study of vertex-transitive tetravalent graphs Λ is to classify these graphs with respect to the permutation group $G_v^{\Lambda(v)}$, induced by the action of G_v on the neighborhood $\Lambda(v)$ of v in Λ (where $G = \mathrm{Aut}(\Lambda)$, or any vertex-transitive subgroup of $\mathrm{Aut}(\Lambda)$). Note that in principle