European Journal of Combinatorics 51 (2016) 306–314

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Small circulant complex Hadamard matrices of Butson type

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ARTICLE INFO

Article history:
Received 7 December 2014
Accepted 19 May 2015
Available online 2 July 2015

ABSTRACT

We study the circulant complex Hadamard matrices of order n whose entries are lth roots of unity. For $n = l$ prime we prove that the only such matrix, up to equivalence, is the Fourier matrix, while for $n = p + q$, $l = pq$ with p, q distinct primes there is no such matrix. We then provide a list of equivalence classes of such matrices, for small values of n, l.

1. Introduction and results

A complex Hadamard matrix of order n is a matrix H having as entries complex numbers of modulus 1, such that $H/\sqrt{n}$ is unitary. Among complex Hadamard matrices, those with all entries roots of unity are said to be of Butson type. The basic example here is the Fourier matrix, $F_n = (w^{ij})_{ij}$ with $w = e^{2\pi i/n}$:

$$F_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{pmatrix}.$$

We denote by $C_n(l)$ the set of complex Hadamard matrices of order n, with all entries being lth roots of unity. As a first example here, observe that we have $F_n \in C_n(n)$.

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http://dx.doi.org/10.1016/j.ejc.2015.05.010
0195-6698/

In general, the complex Hadamard matrices are known to have applications to a wide array of questions, ranging from electrical engineering to von Neumann algebras and quantum physics, and the Butson ones are known to be at the "core" of the theory. For further details here, we recommend the excellent survey article by Tadej and Życzkowski [14], and the subsequent website, made and maintained in collaboration with Bruzda.[1]

We are more specifically interested in understanding the complex Hadamard matrices of Butson type which are *circulant*, that is, of the form $(H_{ij})_{i,j=1,\dots,n}$ with H_{ij} depending only on $i-j$. We denote by $C_n^{circ}(l)$ the set of circulant matrices in $C_n(l)$, and by $C_n^{circ,1}(l)$ the set of matrices from $C_n^{circ}(l)$ with 1 on the diagonal.

Regarding the motivations for the study of such matrices, let us mention: (1) their key importance for the construction of complex Hadamard matrices, see [14], (2) their relation with cyclic n-roots and their applications, see [4], and (3) their relation with the Circulant Hadamard Conjecture, a beautiful mathematical problem, to be explained now.

In the real case – that is, for $l = 2$ – only one example is known, at $n = 4$, namely the 4×4 matrix with -1 on the diagonal and 1 elsewhere. For larger values of n it is conjectured that there is no example:

Conjecture 1.1 (*Circulant Hadamard Conjecture*). *$C_n^{circ}(2) = \emptyset$ for all $n > 4$.*

For larger values of l, however, things are different and, to a large extent, mysterious. Our main goal here is to make some progress on the understanding of the values of n, l which allow the existence of circulant Hadamard matrices of Butson type.

The set of Hadamard matrices is clearly invariant under a number of simple transformations, leading to the following definition. In all the paper "equivalence" between matrices will be understood as defined here.

Definition 1.2. Let M and N be two $n \times n$ complex matrices. We will say that M is *equivalent* to N if it is possible to transform M into N by (i) permuting lines, (ii) permuting columns, (iii) multiplying lines by a constant, (iv) multiplying columns by a constant.

(It should be noted that one could add transposition in this definition, however we prefer not to do this here.)

It is known since [1,7] that F_n is equivalent to a circulant Hadamard matrix, and it follows that $C_n^{circ}(n) \neq \emptyset$. We will prove below that, when n is prime, the Fourier matrix (in circulant form) is the only example, that is, any matrix in $C_n^{circ}(n)$ is equivalent to the Fourier matrix.

Theorem 1.3. *For p prime, any matrix in $C_p^{circ}(p)$ is equivalent to the Fourier matrix F_p.*

Here, as in the rest of the paper, the equivalence relation which is considered among circulant matrices is the cyclic permutation of rows and columns, and multiplication of all entries by a constant. However, since the Fourier matrix is not circulant, we need a wider notion of equivalence in the statement of this theorem. More precisely, we use the standard notion of equivalence for (non-circulant) complex Hadamard matrices, that is, permutation of the rows and columns, and multiplication of each row and each column by a constant, as defined in Definition 1.2.

To put our second statement in perspective, recall that it follows from a result of [11] that if $C_n(p^a q^b) \neq \emptyset$, with p and q prime, then $n \in p\mathbb{N} + q\mathbb{N}$. Moreover we have the following results [2, Theorem 7.9]. Assume $C_n(l) \neq \emptyset$.

(1) If $n = p + 2$ with $p \geq 3$ prime, then l cannot be of the form $2p^b$, with $b \in \mathbb{N}_{>0}$.
(2) If $n = 2q$ with $p > q \geq 3$ primes, then l cannot be of the form $2^a p^b$, with $a, b \in \mathbb{N}_{>0}$.

It follows from the first result that for $p \geq 3$ prime, $C_{p+2}(2p) = \emptyset$, so $C_{p+2}^{circ}(2p) = \emptyset$. However it remains unclear whether $C_{p+3}(3p) = \emptyset$ for $p \geq 5$ prime. It also seems natural to ask whether, for p, q

[1] http://chaos.if.uj.edu.pl/~karol/hadamard.