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Morphic images of episturmian words having finite palindromic defect

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ABSTRACT

We study morphisms from certain classes and their action on episturmian words. The first class is P_{ret}. In general, a morphism of class P_{ret} can map an infinite word having zero palindromic defect to a word having infinite palindromic defect. We show that the image of an episturmian word, which has zero palindromic defect, under a morphism of class P_{ret} has always its palindromic defect finite. We also focus on letter-to-letter morphisms to binary alphabet: we show that images of ternary episturmian words under such morphisms have zero palindromic defect. These results contribute to the study of an unsolved question of characterization of morphisms that preserve finite, or even zero, palindromic defect. They also enable us to construct new examples of binary words having zero or finite H-palindromic defect, where $H = \{\mathrm{Id}, R, E, RE\}$ is the group generated by both involutory antimorphisms on a binary alphabet.

1. Introduction

In combinatorics on words, the most famous class of words probably is the class of Sturmian words: aperiodic words having minimal factor complexity possible (see [24]). Sturmian words are profoundly studied and many generalizations are known, see for instance [4]. One such generalization of Sturmian words are episturmian words. Episturmian words were inspired by Arnoux–Rauzy words (see [28,1]). An infinite word over a k-letter alphabet is *episturmian* if it is closed under reversal and has at most one left special factor of each length. Refer for instance to [17,22,19] for more results on this class.

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A notion related to the study of episturmian words is a *palindrome*—a word equal to its reversal. Episturmian words are rich in palindromes: they contain the maximum number of distinct palindromic factors possible. Precisely, we say that a finite word w is *rich* if it contains exactly $|w| + 1$ distinct palindromic factors, which is the upper bound for the number of distinct palindromic factors in a finite word of length $|w|$ (see [17]). The notion is extended to infinite words: an infinite word is *rich* if every its factor is rich.

In the context of this upper bound on the number of palindromic factors, a measure of the count of missing palindromic factors was introduced in [9]: the *palindromic defect* $D(w)$ of a finite word w is

$$D(w) = |w| + 1 - \#\mathrm{Pal}(w),$$

where $\mathrm{Pal}(w)$ is the set of all palindromic factors of w. The palindromic defect of an infinite word $\mathbf{u}$ is defined by $D(\mathbf{u}) = \sup\{D(w) : w \text{ is a factor of } \mathbf{u}\}$. If $D(\mathbf{u})$ is finite, we say that $\mathbf{u}$ is *almost rich*. (If it is zero, then $\mathbf{u}$ is rich as already mentioned.)

Besides episturmian words, examples of rich words include some well-explored word classes such as words coding symmetric interval exchange transformation (see [2]) and words coding rotation on two intervals (see [8]). Properties and characterizations of rich words are studied for instance in [20,4,13,11]. General properties and characterizations of almost rich words are studied in [20,5,6].

In this paper, we study richness and almost richness of images of episturmian words by a morphism from a specific class. Our first result states that we obtain an almost rich word while using a morphism of class P_{ret} introduced in [5] (see Section 2.2 later for the definition).

Theorem 1. *Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ be an episturmian word and $\pi : \mathcal{A}^* \to \mathcal{B}^*$ be a morphism of class P_{ret}. The word $\pi(\mathbf{u})$ is almost rich.*

The second main result involves a letter-to-letter projection of a ternary episturmian word to a binary alphabet. We use the following definition for such a projection.

Definition 2. Let $\mathcal{A}$ be an alphabet and $\mathcal{A}'$ its proper subset. A morphism $\zeta : \mathcal{A} \to \{A, B\}$ defined by

$$\zeta : a \mapsto \begin{cases} A \text{ if } a \in \mathcal{A}', \\ B \text{ otherwise} \end{cases}$$

is called a *binary projection* from $\mathcal{A}$.

The second main result states that we obtain a rich word by projecting a ternary episturmian word to a binary alphabet.

Theorem 3. *Let $\mathbf{u}$ be an episturmian word over a ternary alphabet $\mathcal{A}$ and ζ be a binary projection from $\mathcal{A}$. The word $\zeta(\mathbf{u})$ is rich.*

Our motivation for these results is to find new binary words which are rich in a generalized sense—with respect to both symmetries given by the involutive antimorphisms on a binary alphabet: the reversal R and the exchange of letters E. We give a definition in Section 4, see also [25,26] for more information on this generalization. To construct new binary words rich in this generalized sense, we use the recent results of [27] which provides theorems that relate the classical richness and the generalized richness on a binary alphabet.

Our computer experiments suggest that we can improve Theorem 3: we can drop the requirement on the size of the alphabet $\mathcal{A}$. We state this hypothesis in the last section along with some comments.

The paper is organized as follows. The next section contains some necessary definitions and basic results. Section 3 contains overview of results on episturmian words and proofs of the main results. Finally, Section 4 contains an application of the main results: a construction of binary words which are rich and almost rich in the generalized sense. The last section states some comments and open questions.