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On isomorphism classes of generalized Fibonacci cubes

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ABSTRACT

The generalized Fibonacci cube $Q_d(f)$ is the subgraph of the d-cube Q_d induced on the set of all strings of length d that do not contain f as a substring. It is proved that if $Q_d(f) \cong Q_d(f')$ then $|f| = |f'|$. The key tool to prove this result is a result of Guibas and Odlyzko about the autocorrelation polynomial associated to a binary string. An example of a family of such strings f, f', where $|f| = |f'| \geq \frac{2}{3}(d+1)$ is found. Strings f and f' with $|f| = |f'| = d - 1$ for which $Q_d(f) \cong Q_d(f')$ are characterized.

1. Introduction

An element of $\{0, 1\}^d$ is called a *binary string* (henceforth just called a *string*) of length d, with the usual concatenation notation. For example, $0^{d-1}1$ is the string of length d consisting of $d - 1$ 0 bits followed by a single 1 bit. We will denote by $e_i = 0^{i-1}10^{d-i}$ the ith unit string in $\{0, 1\}^d$.

Let $d \geq 1$ be a fixed integer. The *d-cube* Q_d is the graph whose vertices are the binary strings of length d, with an edge connecting vertices v_1 and v_2 if the underlying strings differ in exactly one position. Given a graph G, the set of vertices of G is denoted by $V(G)$. We use $d_G(u, v)$ to denote the

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length of a shortest path connecting u and v in G. Lastly, we will write $G \cong H$ to signify that the graphs G and H are isomorphic.

For a given string f and integer d, the *generalized Fibonacci cube* $Q_d(f)$ is the subgraph of Q_d induced by the set of all strings of length d that do not contain f as a consecutive substring. Indeed, this generalizes the notion of the d-dimensional *Fibonacci cube* $\Gamma_d = Q_d(11)$, which is the graph obtained from the d-cube Q_d by removing all vertices that contain the substring 11. Set $n_d(f) = |V(Q_d(f))|$.

Fibonacci cubes were introduced by Hsu [3] as a model for interconnection networks. Like the hypercube graphs, Fibonacci cubes have several properties that make them ideal as a network topology, yet their size grows significantly slower than that of the hypercubes. More precisely, while the hypercube of dimension n has 2^n vertices, the order of Γ_n is asymptotically φ^{n+2}, where φ is the golden ratio. Fibonacci cubes have been extensively investigated; see, for example, the recent survey by Klavžar [7] and even more recent papers of Klavžar and Mollard [8] and Vesel [14]. In the first of these papers, different asymptotic properties of Fibonacci cubes are established, while in the latter a linear recognition algorithm is designed for recognizing Fibonacci cubes, improving the previous best recognition algorithm of Taranenko and Vesel [13].

Later, Ilić, Klavžar, and Rho [4] introduced the idea of generalized Fibonacci cubes (as defined above). Under the same name, the graphs $Q_d(1^s)$ were studied by Liu, Hsu, and Chung [10] and Zagaglia Salvi [16]. The analysis of the properties of generalized Fibonacci cubes led to the study of several problems related to the combinatorics of words. To study their isometric embeddability into hypercubes, good and bad words were introduced by Klavžar and Shpectorov [9], where it was proved that about eight percent of all words are good. Isometric embeddability and hamiltonicity of generalized Fibonacci cubes motivated the ideas of the index and parity of a binary word, as defined by Ilić, Klavžar, and Rho [5,6]. Infinite families of bad strings were found [5,15]. In [1] it was proved that $Q_d(f)$ is 2-connected for any f with $|f| \geq 3$.

In this paper we consider the following fundamental question about the generalized Fibonacci cubes: for which binary strings f and f' and positive integers d are the generalized Fibonacci cubes $Q_d(f)$ and $Q_d(f')$ isomorphic? In the next section, we prove that if $Q_d(f)$ and $Q_d(f')$ have the same order, then the equality $|f| = |f'|$ holds. In addition, a family of strings f, f', with $|f| = |f'| \geq \frac{2}{3}(d+1)$ is found for which $Q_d(f) \cong Q_d(f')$ holds. In the last section, we prove that if $|f| = d - 1$, then $Q_d(f) \cong Q_d(f')$ if and only if f and f' have the same block structure. Several conjectures are posed along the way.

In the rest of the section we introduce additional terminology and notation needed throughout the paper. The complement of a bit x is denoted by $\overline{x}$. It is easy to see that if f' is the binary complement of f, or if f' is the reverse of f (the *reverse* of $f = f_1 \dots f_d$ is $f_d \dots f_1$), then $Q_d(f) \cong Q_d(f')$ for any dimension d. Hence we say that a pair of binary strings f, f' is *trivial* if f' can be obtained from f by binary complementation, reversal, or composition of these mappings. We are therefore only interested in the behavior of the other pairs, which we call the *non-trivial pairs*. A *block* of a binary sting f is a maximal (with respect to inclusion) substring of f consisting of consecutive equal bits. Let $f = 0^{r_1}1^{s_1} \cdots 0^{r_k}1^{s_k}$, where $r_1, s_k \geq 0$, $r_2, \dots, r_k, s_1, \dots, s_{k-1} \geq 1$ and $f' = 0^{r'_1}1^{s'_1} \cdots 0^{r'_\ell}1^{s'_\ell}$, where $r'_1, s'_\ell \geq 0$, $r'_2, \dots, r'_\ell, s'_1, \dots, s'_{\ell-1} \geq 1$ be binary strings. Then f and f' have the same block structure if the following three conditions are satisfied: $k = \ell$, $r_1 = 0$ if and only if $r'_1 = 0$, and $s_k = 0$ if and only if $s'_\ell = 0$.

2. The length of forbidden words

In this section we first prove that a necessary condition for $Q_d(f)$ being isomorphic to $Q_d(f')$ is that $|f| = |f'|$. Then we pose the question whether there is some relation between $|f|$ $(=|f'|)$ and d provided that $Q_d(f) \cong Q_d(f')$. To this end we prove that there exist non-trivial pairs f, f' such that $Q_d(f) \cong Q_d(f')$ and $|f| \geq \frac{2}{3}(d+1)$. We also conjecture that for any non-trivial pair f, f' such that $Q_d(f) \cong Q_d(f')$, we must have $|f| \geq \frac{2}{3}(d+1)$.

The *autocorrelation polynomial* $p_f(z)$ associated to a binary string $f = f_1 \dots f_k \in \{0, 1\}^k$ is defined as

$$p_f(z) = \sum_{i=0}^{k-1} c_i z^i,$$