European Journal of Combinatorics 51 (2016) 380–397

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Strong chromatic index of subcubic planar multigraphs

A.V. Kostochka[a], X. Li[b], W. Ruksasakchai[c], M. Santana[a], T. Wang[d,e,1], G. Yu[f,b]

[a] Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA

[b] Department of Mathematics, Huazhong Normal University, Wuhan, 430079, China

[c] Department of Mathematics, Faculty of Science, KhonKaen University, KhonKaen, 40002, Thailand

[d] Institute of Applied Mathematics, Henan University, Kaifeng, 475004, PR China

[e] College of Mathematics and Information Science, Henan University, Kaifeng, 475004, PR China

[f] Department of Mathematics, The College of William and Mary, Williamsburg, VA, 23185, USA

ARTICLE INFO

Article history:
Received 29 August 2014
Accepted 6 July 2015
Available online 25 July 2015

Dedicated to the memory of Ralph J. Faudree

ABSTRACT

The strong chromatic index of a multigraph is the minimum k such that the edge set can be k-colored requiring that each color class induces a matching. We verify a conjecture of Faudree, Gyárfás, Schelp and Tuza, showing that every planar multigraph with maximum degree at most 3 has strong chromatic index at most 9, which is sharp.

1. Introduction

All multigraphs in this paper are loopless. A *strong k-edge-coloring* of a multigraph G is a coloring $\phi : E(G) \to [k]$ such that if any two edges e_1 and e_2 are either adjacent to each other or adjacent to a common edge, then $\phi(e_1) \neq \phi(e_2)$. In other words, the edges in each color class form an induced matching in the original multigraph. The *strong chromatic index* of G, denoted by $\chi'_s(G)$, is the minimum k for which G has a strong k-edge-coloring. This is equivalent to finding the chromatic number of the square of the line graph of G.

E-mail addresses: kostochk@math.uiuc.edu (A.V. Kostochka), xwli68@mail.ccnu.edu.cn (X. Li), watcharintorn1@hotmail.com (W. Ruksasakchai), santana@illinois.edu (M. Santana), wangtao@henu.edu.cn (T. Wang), gyu@wm.edu (G. Yu).

[1] This research was done while the author was visiting the University of Illinois at Urbana-Champaign.

http://dx.doi.org/10.1016/j.ejc.2015.07.002
0195-6698/

Fouquet and Jolivet [8,7] introduced the notion of strong edge-coloring, which was used to solve a problem involving radio networks and their frequencies. More details on this application can be found in [19,20].

For general graphs, the greedy algorithm provides an upper bound on χ_s' of $2(\Delta-1)+2(\Delta-1)^2+1$, where Δ denotes the maximum degree of the multigraph. At a 1985 seminar in Prague, Erdős and Nešetřil conjectured that in fact a stronger upper bound holds, which if true, is best possible (see [4,5]).

Conjecture 1 (*Erdős and Nešetřil '85*). *If G is a graph with maximum degree Δ, then*

$$\chi_s'(G) \le \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even}, \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{if } \Delta \text{ is odd}. \end{cases}$$

When G has maximum degree at most 3, the conjecture was verified by Andersen [1], who proved the conjecture for multigraphs, and independently by Horák, Qing and Trotter [13]. In general, the problem remains open with the best known upper bound due to Molloy and Reed [17] using probabilistic techniques.[2]

Theorem (*Molloy and Reed '97*). *For large enough Δ, every graph G with maximum degree Δ has $\chi_s'(G) \le 1.998\Delta^2$.*

Faudree et al. [6] show that when restricted to planar multigraphs, $\chi_s'(G) \le 4\Delta + 4\mu$, where μ denotes the maximum number of parallel edges connecting a pair of vertices in G. Additionally, they show that for every positive integer $k \ge 2$, there exists a planar graph G with $\Delta = k$ and $\chi_s'(G) = 4\Delta - 4$.

Borodin and Ivanova [2] show that if a planar graph G has maximum degree at most Δ and girth (i.e. the length of a shortest cycle) at least $40\lfloor \frac{\Delta}{2} \rfloor + 1$, then $\chi_s'(G) \le 2\Delta - 1$.

In regard to *subcubic* graphs, i.e., graphs with maximum degree at most 3, Faudree et al. [6] pose the following set of conjectures.

Conjecture 2 (*Faudree et al. '90*). *Let G be a subcubic graph.*

2.1 $\chi_s'(G) \le 10$.
2.2 *If G is bipartite, then* $\chi_s'(G) \le 9$.
2.3 *If G is planar, then* $\chi_s'(G) \le 9$.
2.4 *If G is bipartite and the degree sum along every edge is at most 5, then* $\chi_s'(G) \le 6$.
2.5 *If G is bipartite with girth at least 6, then* $\chi_s'(G) \le 7$.
2.6 *If G is bipartite with large girth, then* $\chi_s'(G) \le 5$.

Andersen [1], and independently Horák, Qing and Trotter [13], proved Conjecture 2.1. Conjecture 2.2 was verified by Steger and Yu [21]. Conjecture 2.4 was confirmed by Wu and Lin [22] and was generalized by Nakprasit and Nakprasit [18]. The previously mentioned result of Borodin and Ivanova [2] verified Conjecture 2.6 for planar graphs. The authors know of no results which pertain to Conjecture 2.5.

The purpose of this paper is to verify Conjecture 2.3. That is, we prove the following theorem, which is best possible by considering the complement of the cycle of length six.

Theorem 1. *Every subcubic, planar multigraph G with no loops has $\chi_s'(G) \le 9$.*

The proof of this result yields a polynomial time algorithm in terms of the number of vertices that will color any subcubic, planar multigraph using at most nine colors. Theorem 1 implies the following corollary.

[2] Recently, Bruhn and Joos [3] claim to have improved this bound to $1.93\Delta^2$.