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A necessary condition for the tightness of odd-dimensional combinatorial manifolds

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ABSTRACT

We present a necessary condition for $(\ell - 1)$-connected combinatorial $(2\ell + 1)$-manifolds to be tight. As a corollary, we show that there is no tight combinatorial three-manifold with first Betti number at most two other than the boundary of the four-simplex and the nine-vertex triangulation of the three-dimensional Klein bottle.

1. Introduction

Tight combinatorial manifolds are rare but very special objects. There are strong necessary conditions on when a combinatorial manifold can be tight and it is conjectured that all tight combinatorial manifolds are strongly minimal triangulations [15, Conjecture 1.3].

There are two infinite families of tight combinatorial manifolds known due to Kühnel [14] and Datta and Singh [8], and there are a number of very famous sporadic examples, see [15] for an overview. Furthermore, additional examples have been found recently and further infinite families are conjectured [6].

Given a combinatorial manifold M it is difficult to check in general whether or not M is tight. One way to do this is to look at all regular simplex-wise linear functions on M and check if they all have the minimum number of critical points, i.e., if they are all *perfect*, see [2] for an elaborate way to do this. As a consequence, necessary as well as sufficient conditions for tightness are highly sought after.

Here we establish new necessary conditions for the tightness of odd-dimensional combinatorial manifolds by analysing topological properties of slicings, i.e., co-dimension one normal sub-manifolds, which do not depend on the topology of the surrounding manifold.

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As a result, we present upper bounds on the number of vertices of a combinatorial manifold M in terms of its Betti numbers, this way disqualifying large classes of topological manifolds from having tight triangulations at all.

In particular we prove the following result about $(\ell - 1)$-connected combinatorial $(2\ell + 1)$-manifolds.

Theorem 1.1. *Let $\mathbb{M}$ be an $\mathbb{F}$-orientable compact closed $(\ell-1)$-connected $(2\ell+1)$-manifold represented by an n-vertex $\mathbb{F}$-tight combinatorial manifold M. Then*

$$\beta_\ell(\mathbb{M}M, \mathbb{F}) = \beta_{\ell+1}(\mathbb{M}M, \mathbb{F}) \geq \left\lceil (-1)^{\ell+1} \frac{(1 - \lfloor n/2 \rfloor)_{\ell+1} (1 - \lceil n/2 \rceil)_{\ell+1}}{(\ell+1)!\,(1-n)_{\ell+1}} \right\rceil \tag{1.1}$$

where $(a)_n = a \cdot (a+1) \cdot (a+2) \cdot \cdots \cdot (a+n-1)$ denotes the Pochhammer symbol.

Theorem 1.1 complements results about $(\ell - 1)$-connected combinatorial 2ℓ-manifolds due to Kühnel [14]. As of today, the known cases of equality in Inequality (1.1) are the boundary of the simplex ($\ell \geq 1$, $\beta_\ell = 0$) and the 13-vertex triangulation of $SU(3)/SO(3)$ ($\ell = 2$ and $\beta_\ell = 1$).

As a direct consequence any ($\mathbb{F}$-)tight connected combinatorial three-manifold M with $\beta_1(M, \mathbb{F}) \leq 2$ cannot have more than 12 vertices. Together with further results presented in Section 6 and extended computer experiments this leads to the following result.

Corollary 1.2. *The boundary of the simplex and the nine-vertex three-dimensional Klein Bottle $S^2 \underline{\times} S^1$ are the only tight combinatorial three-manifolds with first Betti number at most two.*

The main results of this article can be generalised to homology manifolds: the proof of Theorem 1.1 mainly relies on the Upper Bound Theorem [19,20] and on counting faces. Both of these arguments hold in the more general case of homology manifolds. However, since the main focus of this article is on PL triangulations of manifolds, Theorem 1.1 is phrased in a more specialised form using combinatorial manifolds.

2. Preliminaries

2.1. Combinatorial manifolds

A *combinatorial d-manifold M* is an abstract pure simplicial complex of dimension d such that all vertex links are triangulated standard *PL*-spheres. The *f-vector* of M is a $(d+1)$-tuple $f(M) = (f_0, f_1, \ldots, f_d)$ where f_i denotes the number of i-dimensional faces of M. The zero-dimensional faces of M are called *vertices*, the one-dimensional faces are called *edges* and the d-dimensional faces are referred to as *facets*. The set of vertices of M will be denoted by $V(M)$ or just V if M is given by the context.

We call M *k-neighbourly* if $f_{k-1} = \binom{f_0}{k}$, i.e., if it contains all possible $(k-1)$-dimensional faces. An n-vertex combinatorial d-manifold M distinct from the boundary of the $(d+1)$-simplex can be at most $(\lfloor \frac{d+2}{2} \rfloor)$-neighbourly. In this case the f-vector of an odd-dimensional combinatorial manifold M is already determined to be the one of the boundary complex of the (even-dimensional) cyclic $(d+1)$-polytope with n vertices. This statement is known as the *Upper Bound Theorem* due to Novik [19], and Novik and Swartz [20].

Given a combinatorial manifold M with vertex set $V(M)$ and $W \subset V(M)$, the simplicial complex

$$M[W] = \{\sigma \in M \mid V(\sigma) \subset W\},$$

i.e., the simplicial complex of all faces of M with vertex set in W, is called the *sub-complex of M induced by W*.