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Boxicity and topological invariants

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ABSTRACT

The boxicity of a graph $G = (V, E)$ is the smallest integer k for which there exist k interval graphs $G_i = (V, E_i)$, $1 \leqslant i \leqslant k$, such that $E = E_1 \cap \cdots \cap E_k$. In the first part of this note, we prove that every graph on m edges has boxicity $O(\sqrt{m \log m})$, which is asymptotically best possible. We use this result to study the connection between the boxicity of graphs and their Colin de Verdière invariant, which share many similarities. Known results concerning the two parameters suggest that for any graph G, the boxicity of G is at most the Colin de Verdière invariant of G, denoted by $\mu(G)$. We observe that every graph G has boxicity $O(\mu(G)^4(\log \mu(G))^2)$, while there are graphs G with boxicity $\Omega(\mu(G)\sqrt{\log \mu(G)})$. In the second part of this note, we focus on graphs embeddable on a surface of Euler genus g. We prove that these graphs have boxicity $O(\sqrt{g} \log g)$, while some of these graphs have boxicity $\Omega(\sqrt{g \log g})$. This improves the previously best known upper and lower bounds. These results directly imply a nearly optimal bound on the dimension of the adjacency poset of graphs on surfaces.

1. Introduction

Given a collection $\mathcal{C}$ of subsets of a set Ω, the *intersection graph* of $\mathcal{C}$ is defined as the graph with vertex set $\mathcal{C}$, in which two elements of $\mathcal{C}$ are adjacent if and only if their intersection is non empty. A *d-box* is the Cartesian product $[x_1, y_1] \times \cdots \times [x_d, y_d]$ of d closed intervals of the real line. The *boxicity* $\text{box}(G)$ of a graph G, introduced by Roberts [14] in 1969, is the smallest integer $d \geqslant 1$ such that G is the intersection graph of a collection of d-boxes. The *intersection* $G_1 \cap \cdots \cap G_k$ of k graphs $G_1, \ldots, G_k$ defined on the same vertex set V, is the graph $(V, E_1 \cap \cdots \cap E_k)$, where E_i $(1 \leqslant i \leqslant k)$ denotes the edge

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set of G_i. Observe that the boxicity of a graph G can equivalently be defined as the smallest k such that G is the intersection of k interval graphs.

In the first part of this note, we prove that every graph on m edges has boxicity $O(\sqrt{m \log m})$, and that there are examples showing that this bound is asymptotically best possible.

A minor-monotone graph invariant, usually denoted by $\mu(\cdot)$, was introduced by Colin de Verdière in 1990 [4]. It relates to the maximal multiplicity of the second largest eigenvalue of the adjacency matrix of a graph, in which the diagonal entries can take any value and the entries corresponding to edges can take any positive values (a technical assumption, called the Strong Arnold Property, has to be added to avoid degenerate cases, but we omit the details as they are not necessary in our discussion).

It was proved by Colin de Verdière that $\mu(G) \leqslant 1$ if and only if G is a linear forest, $\mu(G) \leqslant 2$ if and only if G is an outerplanar graph, and $\mu(G) \leqslant 3$ if and only if G is a planar graph. Scheinerman proved in 1984 that outerplanar graphs have boxicity at most two [15] and Thomassen proved in 1986 that planar graphs have boxicity at most three [17]. Since a linear forest is an interval graph, these results prove that for any planar graph G, $\mathrm{box}(G) \leqslant \mu(G)$.

These two graph invariants share several other similarities: every graph G of treewidth at most k has $\mathrm{box}(G) \leqslant k+1$ [3] and $\mu(G) \leqslant k+1$ [9]. For any vertex v of G, $\mathrm{box}(G-v) \leqslant \mathrm{box}(G)+1$ and if $G-v$ contains an edge, $\mu(G-v) \leqslant \mu(G)+1$. Both parameters are bounded for graphs G with crossing number at most k: $\mathrm{box}(G) = O(k^{1/4}(\log k)^{3/4})$ [2] and $\mu(G) \leqslant k+3$ [4]. It is known that every graph on n vertices has boxicity at most $n/2$, and equality holds only for complements of perfect matchings [14]. These graphs have Colin de Verdière invariant at least $n-3$ [11]. On the other hand every graph on n vertices has Colin de Verdière invariant at most $n-1$, and equality holds only for cliques (which have boxicity 1).

It is interesting to note that in each of the results above, the known upper bound on the boxicity is better than the known upper bound on the Colin de Verdière invariant. This suggests that for any graph G, $\mathrm{box}(G) \leqslant \mu(G)$.

The following slightly weaker relationship between the boxicity and the Colin de Verdière invariant is a direct consequence of the fact that any graph G excludes the clique on $\mu(G)+2$ vertices as a minor, and graphs with no K_t-minor have boxicity $O(t^4(\log t)^2)$ [6].

Proposition 1. *There is a constant c_0 such that for any graph G, $\mathrm{box}(G) \leqslant c_0\mu(G)^4(\log \mu(G))^2$.*

It follows that the boxicity is bounded by a polynomial function of the Colin de Verdière invariant.

Pendavingh [13] proved that for any graph G with m edges, $\mu(G) \leqslant \sqrt{2m}$. Interestingly, there did not exist any corresponding result for the boxicity and it was suggested by András Sebő that graphs G with large boxicity (as a function of their number of edges) might satisfy $\mathrm{box}(G) > \mu(G)$. As we observe in the next section, there are graphs on m edges, with boxicity $\Omega(\sqrt{m \log m})$. It follows that there are graphs G with boxicity $\Omega(\mu(G)\sqrt{\log \mu(G)})$. These graphs show that the boxicity is not even bounded by a linear function of the Colin de Verdière invariant.

In the second part of this paper, we show that every graph embeddable on a surface of Euler genus g has boxicity $O(\sqrt{g}\log g)$, while there are graphs embeddable on a surface of Euler genus g with boxicity $\Omega(\sqrt{g \log g})$. This improves the upper bound $O(g)$ and the lower bound $\Omega(\sqrt{g})$ given in [6]. (Incidentally, graphs embeddable on a surface of Euler genus g have Colin de Verdière invariant $O(g)$ and it is conjectured that the right bound should be $O(\sqrt{g})$ [4,16].)

Our upper bound on the boxicity of graphs on surfaces has a direct corollary on the dimension of the adjacency poset of graphs on surfaces, introduced by Felsner and Trotter [8], and investigated in [7] and [6].

2. Boxicity and the number of edges

We will use the following two lemmas of Adiga, Chandran, and Mathew [2]. A graph G is *k-degenerate* if every subgraph of G contains a vertex of degree at most k. In what follows, the logarithm is taken to be the natural logarithm (and its base is denoted by e).

Lemma 2 ([2])**.** *Any k-degenerate graph on $n \geqslant 2$ vertices has boxicity at most $(k+2)\lceil 2e \log n\rceil$.*