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Real zeros and partitions without singleton blocks

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ABSTRACT

We prove that the generating polynomials of partitions of an n-element set into non-singleton blocks, counted by the number of blocks, have real roots only and we study the asymptotic behavior of the leftmost roots. We apply this information to find the most likely number of blocks. Also, we present a quick way to prove the corresponding statement for cycles of permutations in which each cycle is longer than a given integer r.

1. Introduction

A *partition* of the set $[n] = \{1, 2, \ldots, n\}$ is a set of disjoint blocks $B_1, B_2, \ldots, B_k$ so that $\cup_{i=1}^{k} B_i = [n]$. The number of partitions of $[n]$ into k blocks is denoted by $S(n, k)$ and is called a *Stirling number of the second kind*.

Similarly, the number of permutations of length n with exactly k cycles is denoted by $c(n, k)$, and is called a *signless Stirling number of the first kind*. See any textbook on Introductory Combinatorics, such as [1] or [2] for the relevant definitions, or basic facts, on Stirling numbers.

The "horizontal" generating functions, or generating polynomials, of Stirling numbers have many interesting properties. Let n be a fixed positive integer. Then it is well-known (see [1] or [2] for instance) that

$$C_n(x) = \sum_{k=1}^{n} c(n, k)x^k = x(x + 1) \cdots (x + n - 1). \qquad (1)$$

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In particular, the roots of the generating polynomial $C_n(x)$ are all real (indeed, they are the integers $0, -1, -2, \ldots, -(n-1)$).

Similarly, it is known (see [19], page 20, for instance) that for any fixed positive integer n, the roots of the generating polynomial

$$S_n(x) = \sum_{k=1}^{n} S(n,k)x^k$$

are all real, though they are not nearly as easy to describe as those of $C_n(x)$.

Rodney Canfield [5] (in the case of $r = 1$) and Francesco Brenti [4] (in the general case) have generalized (1) as follows. Let $d_r(n,k)$ be the number of permutations of length n that have k cycles, each longer than r. Such permutations are sometimes called r-derangements. Then the generating polynomial

$$d_{n,r}(x) = \sum_{k\geq 1} d_r(n,k)x^k \tag{2}$$

has real roots only. The first author [3] proved that for any given positive integer constant m, there exists a positive number N so that if $n > N$, then one of these roots will be very close to -1, one will be very close to -2, and so on, with one being very close to $-m$, to close the sequence of m roots being very close to consecutive negative integers.

In this paper, we consider the analogue problem for set partitions. Let $D(n,k)$ be the number of partitions of $[n]$ into k blocks, each consisting of more than one element. We are going to prove that the generating polynomial

$$D_n(x) = \sum_{k\geq 1} D(n,k)x^k \tag{3}$$

has real roots only. We will then use this information to determine the location of the largest coefficient (s) of $D_n(x)$. We also prove that the number of blocks is normally distributed. Finally, we use our methods on r-derangements, and prove the more general result that for any fixed r, the distribution of the number of cycles of r-derangements of length n converges to a normal distribution.

Note that the fact that the two kinds of Stirling numbers behave in the same way under this generalization is not completely expected. Indeed, while $1/e$ of all permutations of length n have no cycles of length 1, and in general, a constant factor of permutations of length n have no cycles of length r or less, the corresponding statement is not true for set partitions. Indeed, almost all partitions of $[n]$ contain a singleton block as we show in Section 3.1. However, as this paper proves, the real zeros property survives.

Finally, we mention that the *vertical* generating functions (minimal block or cycle size is fixed, n varies) of permutations and set partitions have been studied in [5].

2. The proof of the real zeros property

We start by a recurrence relation satisfied by the numbers $D(n,k)$ of partitions of $[n]$ into k blocks, each block consisting of more than one element. It is straightforward to see that

$$D(n,k) = kD(n-1,k) + (n-1)D(n-2,k-1). \tag{4}$$

Indeed, the first term of the right-hand side counts partitions of $[n]$ into blocks larger than one in which the element n is in a block larger than two, and the second term of the right-hand side counts those in which n is in a block of size exactly two. We note that this recurrence appears in the classical book of Comtet [8, p. 222].

Let $D_n(x) = \sum_{k\geq 1} D(n,k)x^k$. Then (4) yields

$$D_n(x) = x\left(D'_{n-1}(x) + (n-1)D_{n-2}(x)\right). \tag{5}$$

Note that $D_1(x) = 0$, and $D_n(x) = x$ if $2 \leq n < 4$. So the first non-trivial polynomial $D_n(x)$ occurs when $n = 4$, and then $D_4(x) = 3x^2 + x$. In the next non-trivial case of $n = 5$, we get $D_5 = 10x^2 + x$.