European Journal of Combinatorics 51 (2016) 511–518

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Complex spherical codes with two inner products

Hiroshi Nozaki, Sho Suda

Department of Mathematics Education, Aichi University of Education, Kariya, 448-8542, Japan

ARTICLE INFO

Article history:
Received 12 March 2015
Accepted 22 July 2015
Available online 3 September 2015

ABSTRACT

A finite set X in a complex sphere is called a complex spherical 2-code if the number of inner products between two distinct vectors in X is equal to 2. In this paper, we characterize the tight complex spherical 2-codes by doubly regular tournaments or skew Hadamard matrices. We also give certain maximal 2-codes relating to skew-symmetric D-optimal designs. To prove them, we show the smallest embedding dimension of a tournament into a complex sphere by the multiplicity of the smallest or second-smallest eigenvalue of the Seidel matrix.

1. Introduction

Let X be a finite set of points on the complex unit sphere $\Omega(d)$ in $\mathbb{C}^d$. The *angle set* $A(X)$ is defined to be

$$A(X) = \{x^*y \mid x, y \in X, x \neq y\},$$

where x^* is the transpose conjugate of a column vector x. A finite set X is called a *complex spherical s-code* if $|A(X)| = s$ and $A(X)$ contains an imaginary number. The value s is called the *degree* of X. For $X, X' \subset \Omega(d)$, we say that X is *isomorphic* to X' if there exists a unitary transformation from X to X'. An s-code $X \subset \Omega(d)$ is said to be *largest* if X has the largest possible cardinality in all s-codes in $\Omega(d)$. One of major problems on s-codes is to classify largest s-codes for given s and d.

We will survey Euclidean finite sets with only s distances. For $X \subset \mathbb{R}^d$, we define

$$D(X) = \{d(x, y) \mid x, y \in X, x \neq y\},$$

E-mail addresses: hnozaki@auecc.aichi-edu.ac.jp (H. Nozaki), suda@auecc.aichi-edu.ac.jp (S. Suda).

http://dx.doi.org/10.1016/j.ejc.2015.07.016
0195-6698/

where $d(x, y)$ is the Euclidean distance of x and y. A finite set X is called an *s-distance set* if $|D(X)| = s$ holds. We have an upper bound for the size of an s-distance set in $\mathbb{R}^d$, namely $|X| \leq \binom{d+s}{s}$ [3]. Clearly the largest 1-distance set in $\mathbb{R}^d$ is the regular simplex for any d. Largest 2-distance sets in $\mathbb{R}^d$ are classified for $d \leq 7$ [9,11]. Largest s-distance sets in $\mathbb{R}^2$ are classified for $s \leq 5$ [10,19,20]. The largest 3-distance set in $\mathbb{R}^3$ is the vertex set of the icosahedron [21]. The classification of largest s-distance sets is still open for others (s, d). A largest 2-distance set in $\mathbb{R}^8$ is given in [11], and it attains the upper bound.

A spherical s-distance set particularly deserves attention because of the connection to association schemes or spherical t-designs (see [7,2] for details). A subset X of S^{d-1} is called a *spherical t-design* if for any polynomial f in d variables of degree at most t, the following equality holds:

$$\frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(x)dx = \frac{1}{|X|}\sum_{x\in X} f(x),$$

where $|S^{d-1}|$ is the volume of S^{d-1}. If a spherical t-design X of degree s satisfies $t \geq 2s-2$, then X has the structure of a Q-polynomial association scheme [7]. The size of an s-distance set in S^{d-1} is smaller than or equal to $\binom{d+s-1}{s} + \binom{d+s-2}{s-1}$ [7]. An s-distance set X is said to be *tight* if X attains this bound. A tight s-distance set becomes a minimal spherical t-design and satisfies $t = 2s$ [7]. The classification of tight s-distance sets is one of the most interesting problems, and this has been solved except for $s = 2$ [4]. A largest 2-distance set on S^{d-1} is determined for $d \leq 93$ $(d \neq 46, 78)$ [13,5]. A largest 3-distance set on S^{d-1} is determined for $d = 2, 3, 8, 22$ [21,14].

A simple graph $G = (V, E)$ is *representable* in $\mathbb{R}^d$ if there is an embedding $\sigma : V \to \mathbb{R}^d$ such that

$$d(\sigma(a), \sigma(b)) = \begin{cases} \alpha \text{ if } (a, b) \in E, \\ \beta \text{ otherwise,} \end{cases}$$

for some $\alpha, \beta \in \mathbb{R}$. For a simple graph G, Roy [17] gave an explicit expression of the minimal dimension d such that G is representable in $\mathbb{R}^d$ in terms of the multiplicity of the smallest or second-smallest eigenvalue of A. This embedding of a graph is useful for the classification of 2-distance sets [9,11].

Roy and Suda [18] gave the complex analogue of the spherical s-distance set theory. Complex spherical s-codes are closely related to complex spherical designs or non-symmetric association schemes. In this paper, we consider a complex spherical 2-code $X \subset \Omega(d)$. If X satisfies $A(X) \subset \mathbb{R}$, then the Gram matrix of X is real, and X can be embedded into $\mathbb{R}^d$. We may assume $A(X)$ contains an imaginary number α, and $A(X) = \{\alpha, \overline{\alpha}\}$, where $\overline{\alpha}$ is the conjugate of α. We have a natural upper bound [18]:

$$|X| \leq \begin{cases} 2d+1 & \text{if } d \text{ is odd,} \\ 2d & \text{if } d \text{ is even.} \end{cases} \tag{1.1}$$

A 2-code X is said to be *tight* if X attains the bound (1.1). This is known as the *absolute bound*.

A *tournament* is a directed graph obtained by assigning a direction for each edge in an undirected complete graph. Formally, a tournament is a pair (V, E) such that the vertex set V is a finite set and the edge set $E \subset V \times V$ satisfies $E \cap E^T = \emptyset$ and $E \cup E^T \cup \{(x, x) \mid x \in V\} = V \times V$, where $E^T := \{(x, y) \mid (y, x) \in E\}$. A complex spherical 2-code X has the structure of a tournament (X, E), where $E = \{(x, y) \in X \times X \mid x^*y = \alpha\}$. A tournament (V, E) is *representable in $\Omega(d)$* if there exists a mapping φ from V to $\Omega(d)$ such that for all distinct $x, y \in V$,

$$\varphi(x)^*\varphi(y) = \begin{cases} \alpha \text{ if } (x, y) \in E, \\ \overline{\alpha} \text{ if } (y, x) \in E, \end{cases}$$

where α is an imaginary number with $\mathrm{Im}(\alpha) > 0$. Such a mapping φ is said to be a *representation* of a tournament. We identify a representation with the image of the representation. Two tournaments $G = (V, E), G' = (V', E')$ are *isomorphic* if there is a bijection from V to V' such that $(x, y) \in E$ if and only if $(f(x), f(y)) \in E'$. For two tournaments G and G', if G is not isomorphic to G', then a representation of G is not isomorphic to that of G'. Let $\mathrm{Rep}(G)$ denote the smallest d such that G is representable in $\Omega(d)$. The *Seidel matrix* of G is defined to be $\sqrt{-1}(A - A^T)$, where A is the adjacency