European Journal of Combinatorics 49 (2015) 90–104

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

On decomposing regular graphs into locally irregular subgraphs

O. Baudon [a,b], J. Bensmail [a,b], J. Przybyło [c,1], M. Woźniak [c]

[a] University Bordeaux, LaBRI, UMR 5800, F-33400 Talence, France

[b] CNRS, LaBRI, UMR 5800, F-33400 Talence, France

[c] AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Krakow, Poland

ARTICLE INFO

Article history:
Received 28 September 2013
Accepted 25 February 2015
Available online 21 March 2015

ABSTRACT

A *locally irregular graph* is a graph whose adjacent vertices have distinct degrees. We say that a graph *G can be decomposed into k locally irregular subgraphs* if its edge set may be partitioned into k subsets each of which induces a locally irregular subgraph in G. We characterize all connected graphs which cannot be decomposed into locally irregular subgraphs. These are all of odd size and include paths, cycles and a special class of graphs of maximum degree 3. Moreover we conjecture that apart from these exceptions all other connected graphs can be decomposed into 3 locally irregular subgraphs. Using a combination of a probabilistic approach and some known theorems on degree constrained subgraphs of a given graph, we prove this statement to hold for all regular graphs of degree at least 10^7. We also support this conjecture by showing that decompositions into three or two such subgraphs might be indicated e.g. for some bipartite graphs (including trees), complete graphs and cartesian products of graphs with this property (hypercubes for instance). We also investigate a total version of this problem, where in some sense also the vertices are being prescribed to particular subgraphs of a decomposition. Both the concepts are closely related to the known 1-2-3 Conjecture and 1-2 Conjecture, respectively, and other similar problems concerning edge colourings. In particular, we improve the result of Addario-Berry et al. (2005) in the case of regular graphs.

E-mail addresses: olivier.baudon@labri.fr (O. Baudon), jbensmai@labri.fr (J. Bensmail), jakubprz@agh.edu.pl (J. Przybyło), mwozniak@agh.edu.pl (M. Woźniak).

[1] Tel.: +48 12 617 46 38.

http://dx.doi.org/10.1016/j.ejc.2015.02.031
0195-6698/

1. Introduction

All graphs considered are simple and finite. We follow [10] for the notations and terminology not defined here. Consider a graph $G = (V, E)$. It is well known that if its order n is at least two, then it cannot be (completely) *irregular*, i.e., it must contain a pair of vertices of the same degree. By a *locally irregular graph* we shall mean a graph such that the degree of every vertex is distinct from the degrees of all of its neighbours. In other words, it is a graph in which the adjacent vertices have distinct degrees. Such graphs exist for every order n. A natural antonym of the class of these is the family of regular graphs. In this paper we investigate decompositions of regular, or more generally any graphs into locally irregular subgraphs. More precisely, we say that G can be *decomposed into k locally irregular subgraphs* if its edge set may be partitioned into k subsets each of which induces a locally irregular subgraph, i.e., $E = E_1 \cup E_2 \cup \cdots \cup E_k$ with $E_i \cap E_j = \emptyset$ for $i \neq j$ and $H_i := (V, E_i)$ is locally irregular for $i = 1, 2, \ldots, k$. Note that instead of decomposing the graph G, we may alterably paint its edges with k colours so that every colour class induces a locally irregular subgraph in G. Such colouring shall be called a *locally irregular k-edge colouring* of G. The colour classes of this naturally define locally irregular subgraphs of G making up its decomposition. Thus the two notions shall be used equivalently. Note that if an edge $uv \in E$ has colour i assigned by a locally irregular edge colouring, then the number of edges coloured with i incident with u must be distinct from the number of edges coloured with i incident with v. As usual we shall be most interested in the least number of colours necessary to create such a colouring. However, not every graph admits any such colour assignment, this does not exist e.g. for the path P_2 (on 2 vertices). Other exceptions are discussed further on. Apart from these, we suspect that three colours ($k = 3$) are sufficient for all remaining graphs, cf. Conjecture 3.4. Intriguingly, the subject of our investigations binds several other related problems, which in fact motivated our research.

1.1. 1-2-3 Conjecture

Consider another concept of introducing local irregularity in a graph by means of edge colourings (or *weightings*). Let $c : E \to \{1, 2, \ldots, k\}$ be an edge colouring of G with positive integers. For every vertex v we then denote by $s_c(v) := \sum_{u \in N(v)} c(uv)$ the sum of its incident colours and call it the *weighted degree* of v. We say that c is a *neighbour sum distinguishing k-edge colouring* of G if $s_c(u) \neq s_c(v)$ for all adjacent vertices u, v in G. Another interpretation of this concept, introduced by Karoński, Łuczak and Thomason [17], asserts that instead of assigning an integer to every edge, we multiply it the corresponding number of times in order to create a *locally irregular multigraph* of G, i.e., a multigraph whose neighbours have distinct degrees. This problem came to life as a descendant of the graph invariant known as the *irregularity strength* of a graph, where as above, given a graph, one strives to create of it a multigraph in which *all* vertices have distinct degrees, see e.g. [4,9,11,16,18,19,21] for further details and some of the most up-to-date results and open problems concerning this parameter. It is also worth mentioning that the concept of the irregularity strength was motivated by the study of Chartrand, Erdős, Oellermann et al. concerning 'irregular graphs' (see [5,6,8]), whose research are also closely related to ours.

In [17] Karoński, Łuczak and Thomason posed the following elegant problem, known as the *1-2-3 Conjecture*.

Conjecture 1.1 (*1-2-3 Conjecture*). *There exists a neighbour sum distinguishing* 3*-edge colouring of every graph G containing no isolated edges.*

Thus far it is known that a neighbour sum distinguishing 5-edge colouring exists for every graph without isolated edges, see [15]. On the other hand, Addario-Berry, Dalal and Reed proved in [3] the following result for random graphs.

Theorem 1.2. *If G is a random graph (chosen from $G_{n,p}$ for a constant $p \in (0, 1)$), then asymptotically almost surely, there exists a neighbour sum distinguishing* 2*-edge colouring of G.*

This fact gets even more interesting in view of our research if combined with the following straightforward observation.